The Foundations of Quantum Information and Feasible Experiments

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Contents

Abstract

This thesis contains results on different questions in quantum information. It consists of four chapters. The subject of the first chapter is the copying of quantum states by stimulated emission. According to the no-cloning theorem by Wootters and Zurek[92] it is fundamentally impossible to build a machine which would be able to produce an exact copy of a quantum system in an unknown state. The impossibility of perfect copying follows immediately from the linearity of quantum physics. Approximate copying however is compatible with the principles of quantum mechanics. Quantum mechanics only gives bounds on the fidelity of the copies. Stimulated emission, which is at the heart of the laser, is a natural candidate for the practical realization of a quantum copier. Here it is shown that optimal (i.e. saturating the quantum mechanical bounds) copying of photons can be realized by stimulated emission in simple quantum optical systems, for example three-level atoms. The fidelity of the copies is limited by the unavoidable presence of spontaneous emission, which thus assures that the quantum mechanical bounds are obeyed.

In spite of its non-local features such as the violation of Bell's inequalities, quantum physics is entirely compatible with the special theory of relativity. In particular, entangled states cannot be used for superluminal communication. This peaceful coexistence has led to the question whether the impossibility of superluminal signaling could be used as an axiom in deriving basic features of quantum mechanics from fundamental principles. In the second chapter we show that this is indeed the case. If the kinematical features of quantum physics including the projection postulate are assumed to be given, then its dynamical rules can be derived with the help of the no-signaling condition. This also puts constraints on possible non-linear modifications of quantum mechanics.

Quantum mechanics usually only allows statistical predictions for the behavior of individual physical systems. The third and fourth chapter of this thesis are devoted to theorems on the existence of hidden variables which would make it possible to make predictions for individual systems. One of the classical hidden-variable theorems is the one by Kochen and Specker, which states that so called non-contextual hidden variables are incompatible with quantum mechanics. A new, much simplified, version of this theorem is given, which leads to a proposal for a simple experimental test of non-contextual hidden variables, for example with single photons and linear

optical elements.

The fourth chapter treats the derivation of hidden-variable theorems for real experiments, in particular for finite measurement precision. This investigation was motivated by recent claims that the Kochen-Specker theorem loses its validity under such conditions. It is shown that the basic statements of hidden-variable theorems are robust under real-world conditions.

Preface

Let me begin with some remarks on how I ended up doing precisely the things which are collected in this thesis, and not something else. The emphasis of my undergraduate studies in Vienna and then also in Paris was on theoretical particle physics. I had chosen this subject at the beginning of my studies because it seemed the most fundamental area of physics.

But already as an undergraduate I became very fascinated by the mysterious features of quantum mechanics. I remember that I first heard about Bell's inequalities from Robin Michaels, then a mathematics student in Cambridge, shortly after beginning my studies. I already knew the basic principles of quantum mechanics at that time, but I had had a naive realistic view concerning its statistical predictions, as I realized through our discussion. Later, I attended the seminar on the Foundations of Quantum Mechanics organized by Reinhold Bertlmann and Anton Zeilinger, which was my first contact with Anton.

Towards the end of my studies I realized that I wanted to learn more about the fundamental questions of quantum mechanics, and, if possible, work on them. I thought that the most fundamental question was whether there is something beyond quantum mechanics, or whether we have to content ourselves with its highly idiosyncratic ways of giving us information about the world. I was aware that I would most probably not be able to answer this question during my PhD, but it certainly was the guiding star of my decision. I was also aware that the question is an experimental one. Up to this point, I had not learned very much about experiments.

Given all this, it was quite natural that I joined Anton's group in Innsbruck in December 1997. It is worth mentioning that I arrived there more or less simultaneously with the first TV crews wanting to know about teleportation. I was determined to learn as much as possible about the experimental side. I was lucky enough to spend the first year working as an apprentice on the up to now best experimental test of Bell's inequalities, Gregor Weihs' PhD experiment. Although my time as an experimentalist was actually not very long, it was a very valuable experience. I am convinced that it made me a better physicist, also in theory.

I had always intended to do some theoretical work on the side. Our move from Innsbruck to Vienna created some additional spare time. Both my work on cloning

and on a Kochen-Specker experiment was started in 1998, while we were still in Innsbruck. Anton proposed a Kochen-Specker experiment as a possible topic for my thesis shortly after my arrival in Innsbruck, and he also suggested the work of Cabello and García-Alcaine as a starting point. I still remember the first discussions with Marek Żukowski, Harald Weinfurter and Anton quite vividly. We finished this project much later, in March 2000.

My interest in cloning was triggered at a European Quantum Information meeting in Helsinki. Gregor told me that in Innsbruck they had been discussing the relation of cloning and stimulated emission before, which started our common work on this subject. I remember that from the beginning Anton was interested in the relation between cloning and superluminal communication. The year after that we continued (and in a sense completed) our work together with Julia Kempe.

During my undergraduate studies I had not heard much about the new field of quantum information. I remember reading an introduction to quantum computing (by Adriano Barenco) in Paris, which I had downloaded from the quant-ph folder. In Innsbruck, somewhat unexpectedly, I found myself in one of the centers of the new field. There was not only our group but also our theoretical and experimental colleagues, with Peter Zoller, Ignacio Cirac and Rainer Blatt, so there was a lot to learn.

Today, when asked what I do, I often call myself a theorist in quantum information, and I will soon be a postdoc in a "centre for quantum computation". I have remained true to my foundational interests, as illustrated by my work with Časlav Brukner and Anton on hidden-variable theorems for real experiments, and my work with Vladimir Bužek and Nicolas Gisin on the no-signaling condition. It was of course very helpful to have a boss like Anton, who is himself so deeply fascinated by quantum physics. I am curious to see which unexpected turns the future will bring. Meanwhile I hope that the products of my efforts collected here will be of interest to some of my colleagues.

1 Cloning via Stimulated Emission

1.1 Introduction – Quantum Information

The first part of this thesis is concerned with quantum cloning, in particular with the realization of quantum cloning using stimulated emission. As our work on this topic was usually published under the heading of quantum information in the respective journals it seems appropriate to start with a few remarks on quantum information in general [12].

What is quantum information? Let me try to give two tentative definitions, both formulated as questions. The first one may seem rather too broad, the second one rather too narrow. The first runs as follows: what can you do with quantum mechanics that you cannot do classically? This describes the spirit of the field rather well, but it doesn't quite explain the name quantum information. The second runs: what happens if one has qubits instead of classical bits? As a definition of quantum information this is certainly too restricted, but some of its proudest achievements, such as the celebrated quantum computing algorithms, fit very well into this framework. Furthermore, this definition clearly emphasizes the information processing aspect of the field.

Nowadays, everybody knows what a bit is. Physically a bit is represented by a system with two possible states which are clearly distinguishable, conventionally denoted as 0 and 1. There are of course many possible physical implementations, from smoke signals over pulses of voltage to zones of magnetization on a hard disk. A common feature of all these implementations is that the system can always be determined to be in one of the two relevant states, 0 or 1. If this is not possible, we are dealing with a bad implementation. Now consider a quantum system, which can also be in two clearly distinguishable, i.e. orthogonal states, $|0\rangle$ and $|1\rangle$. Then it follows from the basic principles of quantum mechanics that all properly normalized superposition states

$$\alpha|0\rangle + \beta|1\rangle, \text{ with } |\alpha|^2 + |\beta|^2 = 1 \tag{1.1}$$

are also possible physical states of the system. In the following we will denote such

two-dimensional systems as qubits.

The distinction between qubits and classical bits becomes even more pronounced when several systems are considered. Two classical bits can be in 4 different states: 00, 01, 10 and 11. For qubits any linear combination of the corresponding basis states, $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$, corresponds to a possible state of the physical system. This means that the two qubits can also be in entangled states, such as

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{1.2}$$

Entangled states have been known for a long time to exhibit phenomena which are entirely incompatible with the world-view of classical physics, such as the violation of Bell's inequalities [96, 7]. It may therefore not seem too surprising that replacing bits by qubits can lead to rather different new results.

The most important theoretical developments in the field of quantum information have been the discovery of quantum cryptography and of the quantum computing algorithms by Shor [83] and Grover [48]. Quantum cryptography [11] establishes an entirely secure communication channel between two distant parties. It is based on the fact that in quantum mechanics in general there is no way of performing a measurement without disturbing the state of the system. This implies that in an appropriately designed scheme, any eavesdropper trying to listen in will always be detected. There is no parallel to quantum cryptography in the realm of classical physics. In the past few years there have been many quantum cryptography experiments of increasing practicality and sophistication [54, 71, 87]. Of all the quantum information paradigms, quantum cryptography has certainly come closest to being a usable technology.

The first milestone in the field of quantum computation, i.e. computation based on qubits instead of classical bits, was Shor's discovery of his factoring algorithm [83]. Shor showed that on a quantum computer it is possible to factor large numbers by performing a number of operations that scales only polynomially with the size of the input, whereas the fastest known classical algorithms require an exponential number of steps. This was the first serious indication that in the long run quantum computers may be dramatically faster than classical computers. Since then a lot of work has been invested in trying to find other relevant algorithms which show a similar exponential speedup. So far these attempts have not been successful. However, Grover [48] succeeded in showing that quantum computers also out-perform classical computers in solving a very common task, namely in searching a completely unordered database - imagine for example that you are given a phone number and telephone directory and you want to find out which name the number belongs to. Although the speedup is less dramatic in this case than in the case of factoring,

Grover's discovery may well turn out to be of great practical importance. It should be noted, however, that it requires the database to be given in the form of a quantum mechanical superposition state.

As the existence of entangled states is one of the most important differences between quantum and classical mechanics, it is not too surprising that in both these quantum algorithms entanglement plays a decisive role.

As for the practical implementation of quantum computing, there is still a long way to go. However, simple algorithms using a small number of qubits have been realized in various physical systems, such as in nuclear magnetic resonance [32], cavity QED [88, 49] and ion traps [26, 69]. It is fair to say that nobody currently knows whether it will be possible to build a large-scale quantum computer working with hundreds or thousands of qubits. The main practical difficulty is decoherence [98]. Under normal circumstances, a multi-qubit entangled state would immediately be destroyed by the interaction with its environment. It is virtually impossible to shield a quantum computer from its environment to such a degree as to prevent this from happening. However, there is hope, based on the development of quantum error correction [81, 25, 86] and fault tolerant quantum computation [82]. Using methods that are similar to classical error-correction techniques, but much more subtle, it is possible to detect and correct errors occurring during the quantum computation, including those caused by decoherence. However, these schemes already presume the existence of quantum computer elements which work quite reliably, that is, where the error probabilities are below certain threshold values, typically far below the percent level (per operation).

Let us now come back to the differences between classical bits and qubits. One very important difference is the following: The state of an unknown bit is very easy to determine because it can only be 0 or 1. On the other hand it is completely impossible to determine the state of an unknown qubit if one is given only a single copy of the system. This is easy to see: the unknown qubit could be in any superposition state $\alpha|0\rangle + \beta|1\rangle$ and it turns out that the best thing that the observer can do is to perform a projective measurement in some basis of the Hilbert space spanned by the states $|0\rangle$ and $|1\rangle$ [62]. Such a measurement only gives him one bit of information while he would need an infinity of bits to exactly determine α and β. Interestingly, it is possible to teleport an unknown qubit to a distant location with the help of quantum entanglement [10, 13]. This can be done without finding out anything about the qubit's state.

A related distinction between classical bits and qubits is the impossibility of copying the latter. It is very easy to copy a classical bit: even if it is originally unknown, one simply has to determine its state, 0 or 1, and then produce one more bit in the desired state. Our discussion above indicates that such an approach cannot work for

qubits. Actually there is no way of constructing a perfect quantum copying machine, that is, a machine which given a qubit in an unknown state produces a copy. This is the content of the famous quantum no–cloning theorem [92, 37]. In the following we will see that the impossibility of copying quantum information has deep roots: it is related to the linearity of quantum mechanics, which is in turn related to the impossibility of superluminal communication.

1.2 Signaling and Cloning

To our knowledge, the discussion about the cloning of quantum systems started with a paper by Herbert [50], where he proposed a method for superluminal communication. His scheme made use of pairs of entangled particles shared by the two parties that would like to communicate (Alice and Bob), and of what he called idealized laser tubes, which would today be called universal cloning machines. The basic idea of his proposal was the following. Alice and Bob each have one member of a pair of entangled particles, e.g. photons described by the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|V_A H_B\rangle - |H_A V_B\rangle), \tag{1.3}$$

where V and H denote vertical and horizontal polarization respectively. Alice can measure the polarization of her particle either in the basis $|V\rangle, |H\rangle$ or in the basis $|P\rangle, |M\rangle$, where $|P\rangle = \frac{1}{\sqrt{2}}(|V\rangle + |H\rangle)$ and $|M\rangle = \frac{1}{\sqrt{2}}(|V\rangle - |H\rangle)$. If Alice measures in the V/H basis and finds $|V\rangle$ ($|H\rangle$), Bob's photon is reduced to $|H\rangle$ ($|V\rangle$), while if she measures in the P/M basis (results $|P\rangle$ and $|M\rangle$) Bob's photon is reduced to the corresponding states in that basis ($|M\rangle$ and $|P\rangle$ respectively). Although the states on his side are therefore different depending on Alice's choice of basis, a priori this does not allow Bob to know her choice because he cannot discriminate $|V\rangle$ from $|P\rangle$ by a single measurement (having only a single copy). But imagine that he has a machine that can produce an arbitrary number of copies of any one-photon state, or at least of the states $|V\rangle$ and $|P\rangle$. This would allow Bob to discriminate the two states and in this way Alice's two choices of basis. If Bob's copier works fast enough, this establishes a superluminal communication channel. Herbert proposed stimulated emission as a possible working principle for his copying machine. He was aware of the fact that spontaneous emission could be a problem, but thought that it would not be fatal for the scheme.

On the other hand, it is possible to show in a general way that superluminal signaling (or, more precisely, signaling between systems whose operator algebras commute) is not possible in quantum mechanics [40]. One way of explaining why Alice cannot signal to Bob is the following. Bob does not know which result Alice got, therefore he

has to trace over her degrees of freedom. Thus his photon is described by a density matrix $\frac{1}{2}(|V\rangle\langle V| + |H\rangle\langle H|)$, if she measured in the V/H basis, and $\frac{1}{2}(|P\rangle\langle P| + |M\rangle\langle M|)$, if she measured in the P/M basis. Of course, these two density matrices are identical, so there is no way for him to tell what she did. Now it is clear that no quantum–mechanical device can lead to a distinction between identical density matrices. So something has to be wrong with Herbert's argumentation. Cf. also our discussion in the second chapter of this thesis.

Wootters and Zurek [92] and Dieks [37] showed that the problematic part of Herbert's proposal is the copying procedure. To see this consider a device which produces perfect copies of V and H polarized photons, i.e. which performs the following unitary transformation:

$$\begin{aligned} |V\rangle|\psi_0\rangle &\to |V\rangle|V\rangle|\psi_V\rangle \\ |H\rangle|\psi_0\rangle &\to |H\rangle|H\rangle|\psi_H\rangle \end{aligned} \tag{1.4}$$

How does such a device act on a photon which is linearly polarized under 45 degrees, i.e. described by a state vector $|P\rangle = \frac{1}{\sqrt{2}}(|V\rangle + |H\rangle)$? From a perfect copier we would expect an output of the form $|P\rangle|P\rangle|\psi_P\rangle$. On the other hand it follows from Eq. (1.4) that

$$\frac{1}{\sqrt{2}}(|V\rangle + |H\rangle)|\psi_0\rangle \to \frac{1}{\sqrt{2}}\left(|V\rangle|V\rangle|\psi_V\rangle + |H\rangle|H\rangle|\psi_H\rangle\right) \tag{1.5}$$

It is not hard to see that this output will never be of the desired form. If $|\psi_V\rangle \neq |\psi_H\rangle$, the state of the two copies will even be mixed, which is certainly not what we want. If $|\psi_V\rangle = |\psi_H\rangle$, the state of the two copies will be given by $\frac{1}{\sqrt{2}}\left(|V\rangle|V\rangle + |H\rangle|H\rangle\right)$ which is different from the desired state $|P\rangle|P\rangle = \frac{1}{2}(|V\rangle + |H\rangle)(|V\rangle + |H\rangle)$. This shows that a perfect cloner of photons in the V/H Basis is a maximally bad cloner for states in the complementary basis. Perfect cloning of general input states is impossible. This is the famous no–cloning theorem. It was also pointed out by Mandel [61] and Milonni and Hardies [68], that perfect cloning in stimulated–emission schemes such as proposed by Herbert is prevented by the unavoidable presence of spontaneous emission. This will be discussed in detail in the sequel. With all these results the discussion on cloning was closed for about fourteen years.

In 1996 Bužek and Hillery [21] considered cloning from a different point of view. They proposed and studied an approximate copying machine, i.e. a device which, given a qubit in an unknown input state, produces two approximate copies. They also demanded that their machine be universal, i.e. the quality of the copies should be the same for all inputs. Bužek and Hillery showed that such a universal quantum copying machine can be described by the following unitary transformation:

$$|0_a\rangle|0_b\rangle|0_c\rangle \to \sqrt{\frac{2}{3}}|0_a\rangle|0_b\rangle|1_c\rangle + \sqrt{\frac{1}{6}}(|0_a\rangle|1_b\rangle + |1_a\rangle|0_b\rangle)|0_c\rangle$$

$$|1_a\rangle|0_b\rangle|0_c\rangle \rightarrow \sqrt{\frac{2}{3}}|1_a\rangle|1_b\rangle|0_c\rangle + \sqrt{\frac{1}{6}}(|0_a\rangle|1_b\rangle + |1_a\rangle|0_b\rangle)|1_c\rangle \tag{1.6}$$

where a is the system to be copied, b is an auxiliary qubit which becomes a copy of a through the transformation, and c is another auxiliary qubit. Several remarks are in order. The final state of Eq. (1.6) is invariant under the exchange of a and b, i.e. the copying machine produces two copies with exactly the same properties. In this respect, the distinction between the original qubit and its copy is completely lost during the copying procedure. The quality of the copies can be quantified by the fidelity with respect to the input state, i.e. by $F = \langle\psi|\rho_a|\psi\rangle$, where $|\psi\rangle$ is the state of the original qubit and $\rho_a = \mathrm{Tr}_{bc}\rho_{abc}$ is the reduced density matrix of the qubit a in the final state ρ_{abc}. Applying this formula to Eq. (1.6) gives $F = \frac{5}{6}$. One can show that the transformation (1.6) is indeed universal, i.e. any arbitrary input state $\alpha|0\rangle + \beta|1\rangle$ is copied with the same fidelity. In their seminal paper Bužek and Hillery did not show that their copying machine is optimal.

The Bužek-Hillery construction was generalized by Gisin and Massar [44], who found a cloning transformation producing M copies starting from N identical qubits. It is described by

$$\begin{aligned} U_{N,M}|N_\psi\rangle &= \sum_{j=0}^{M-N} \alpha_j |(M-j)\psi, j\psi^\perp\rangle \otimes R_j(\psi), \\ \alpha_j &= \sqrt{\frac{N+1}{M+1}}\sqrt{\frac{(M-N)!(M-j)!}{(M-N-j)!M!}} \end{aligned} \tag{1.7}$$

where $|N\psi\rangle$ is the input state consisting of N qubits all in the state ψ, we have denoted $|(M-j)\psi, j\psi^\perp\rangle$ the symmetric and normalized state with $M-j$ qubits in the state ψ and j qubits in the orthogonal state $\psi^\perp$. $R_j(\psi)$ are orthogonal states of the ancillary qubits which can be written as $R_j(\psi) = |(M-1-j)\psi^*, j(\psi^*)^\perp\rangle$, where ψ^* is the complex conjugate of ψ.

The transformation (1.7) looks rather intimidating, or at least unintuitive. We will see that actually it arises quite naturally in the context of stimulated emission for appropriately designed systems. The form of the coefficients α_j was also already explained in a rather intuitive way by the work of Werner [91], which we will discuss below. The optimality of the Bužek-Hillery and Gisin-Massar transformations was first shown by Bruß and co–workers in [18], who made use of the relations between quantum cloning and the estimation of quantum states. The general idea of this approach is the following: One way of estimating an unknown quantum state is to first clone it and then perform a state estimation of its clones. But this cannot be better than the optimal state estimation, whose fidelity is known for qubits. On the other hand, one way of cloning an unknown state is to first estimate it and then produce as many copies as desired of the estimated state. But this cannot be better

than optimal cloning. Using these two relations, Bruß et al. derived bounds on the optimal cloning fidelities which are saturated by the Bužek-Hillery and Gisin-Massar transformations. The optimal fidelity for the case of N to M cloning of qubits is

$$F_{N\to M} = \frac{NM + M + N}{M(M+2)} \tag{1.8}$$

It is not hard to see that the Gisin–Massar transformation (1.7) has this copying fidelity. In this case the fidelity can be expressed as the mean relative frequency of qubits in the original state ψ in the final state, i.e.

$$F = \sum_{j=0}^{M-1} \frac{M-j}{M} \alpha_j^2 = F_{N\to M}, \tag{1.9}$$

as can be confirmed using the explicit form of the coefficients α_j. The Bužek–Hillery transformation is contained in the Gisin–Massar transformation as the simplest special case. As mentioned before it leads exactly to $F = F_{1\to 2} = \frac{5}{6}$.

Bužek, Hillery and Werner showed that by choosing the states $R_j(\psi)$ in the Gisin–Massar transformation (1.7) appropriately, it is possible to realize optimal universal cloning and the optimal universal NOT operation simultaneously [19]. The ideal universal NOT would be an operation that produces the orthogonal complement of an arbitrary qubit. Like perfect cloning, this is prohibited by quantum mechanics.

The understanding of the structure of the cloning transformations was significantly deepened by the work of Werner, who proved that the final density matrix of the clones in the optimal N to M cloning transformation can be found in the following way, apart from normalization:

$$\rho_M = P_M^+ (\sigma^{\otimes N} \otimes \mathbb{1}^{\otimes (M-N)}) P_M^+ \tag{1.10}$$

where $\sigma = |\psi\rangle\langle\psi|$, is the original state of the N initial qubits, $\mathbb{1}$ is the completely mixed density matrix, and P_M^+ is the projector onto the completely symmetric subspace of the N–qubit Hilbert space. This is a rather intuitive formula. One can say that at the beginning all the information is contained in the N original qubits. The $M-N$ auxiliary qubits are completely mixed and thus contain no information. Then, the information is distributed over all M qubits in a completely symmetric way. The expression (1.7) for the coefficients α_j can be easily derived from Eq. (1.10).

1.3 Optimal Universal Cloning Transformations

The consideration of Werner's formula Eq. (1.10) leads to a better understanding of the Bužek–Hillery and Gisin–Massar transformations. Let us start with the case

of $1 \to 2$ cloning of qubits. We look for a universal transformation producing two approximate copies of the state ψ. It is clear that ancillas may be needed. What are the basic elements out of which the final state could be built? If, for the moment, we take universality to mean that no direction may be singled out by the form of the final state, apart from the direction of ψ, then it is clear that the only elements permitted are ψ itself, which may appear only linearly (because of the linearity of quantum mechanics), and the singlet state of two qubits $S = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle - |1\rangle|0\rangle)$. The singlet can be rewritten as $S = \frac{1}{\sqrt{2}}\left(\psi\psi^\perp - \psi^\perp\psi\right)$, where $\psi^\perp = -\beta^*|0\rangle + \alpha^*|1\rangle$ for $\psi = \alpha|0\rangle + \beta|1\rangle$. The most general form of the final state is then the linear combination

$$A\,\psi_1\,S_{23} + B\,\psi_2\,S_{13}. \tag{1.11}$$

Note that $S_{12}\psi_3$ is not linearly independent. Eq. (1.11) can be rewritten as

$$A\psi_1\left(\psi_2\psi_3^\perp - \psi_2^\perp\psi_3\right) + B\left(\psi_1\psi_2\psi_3^\perp - \psi_1^\perp\psi_2\psi_3\right) = (A+B)\psi\psi\psi^\perp - \left(A\psi\psi^\perp + B\psi^\perp\psi\right)\psi. \tag{1.12}$$

This manifestly corresponds to a universal transformation because everything is expressed in terms of ψ and $\psi^\perp$. Its linearity is obvious by construction. The case $A = B$ is exactly the Bužek–Hillery symmetric universal cloner. The cases $A \neq B$ correspond to asymmetric universal cloners [70, 15]. The third particle is an anti–clone.

As a first generalization, the above construction can be extended to the $N \to M$ cloning case. Now linearity implies that N instances of ψ have to appear in the final state, which have to be supplemented by $M - N$ singlets. Thus, the final state is a linear combination of the term

$$\psi_1 \ldots \psi_N\, S_{N+1,M+1} \ldots S_{M,2M-N} \tag{1.13}$$

and its permutations. The Gisin–Massar optimal universal cloners correspond to the case where the above expression is symmetrized over the first N qubits. This gives an even more precise sense to our above statement that the information about ψ is spread out over the M copies in a symmetric way. Originally, the qubits $N+1, \ldots, M$ contain no information because they belong to singlet states. Not completely symmetrized linear combinations correspond to general asymmetric cloners.

To generalize the above considerations to the d–dimensional case, where there are no singlet states, one has to note that our above definition of universality was more restrictive than the usual one. Let us now only demand that the reduced density matrix of each clone be of the form $s|\psi\rangle\langle\psi| + \frac{(1-s)}{d}\mathbb{1}$. Then one can replace the singlets in the above construction by the maximally entangled state $X = \sum_{n=1}^{d} |n\rangle|n\rangle$. Apart from this substitution the calculation remains exactly the same as for qubits. In the

$1 \to 2$ case, one has the family of final states

$$A\psi_1 X_{23} + B\psi_2 X_{13} = A\sum_{n=1}^{d} |\psi\rangle|n\rangle|n\rangle + B\sum_{n=1}^{d} |n\rangle|\psi\rangle|n\rangle. \tag{1.14}$$

To determine the reduced density matrix of the first particle, consider the density matrix of the first two:

$$\sum_{n=1}^{d} \left(A|\psi\rangle|n\rangle + B|n\rangle|\psi\rangle\right)\left(A^*\langle\psi|\langle n| + B^*\langle n|\langle\psi|\right). \tag{1.15}$$

Tracing over the second particle, one sees that the reduced density matrix indeed only depends on $|\psi\rangle\langle\psi|$ and $\mathbb{1}$. Again the case $A = B$ corresponds to the general universal optimal cloners of Bužek and Hillery [22]. The construction generalizes to the $N \to M$ case [23]. Thus we have derived all the optimal universal cloning transformations in a simple and unified way.

We will come back to the above considerations when explaining why our stimulated emission cloners are optimal in section 1.7. We now turn to a detailed investigation of quantum cloning by stimulated emission. The relationship between cloning and superluminal communication suggested by Herbert's original proposal will be explored in the sequel.

1.4 Cloning with Lambda–Atoms

When the Bužek–Hillery and Gisin–Massar transformations were discovered, the realization was first discussed in the context of quantum computation, in other words in terms of gates. For example, a network for the Bužek–Hillery cloner was suggested in [20]. However, there is actually a physical process which seems a very natural candidate for quantum cloning, namely stimulated emission. This was already realized by Herbert [50], who thought however about perfect cloning. It was then pointed out by Mandel [61] and Milonni and Hardies [68] that perfect cloning is prevented by the presence of spontaneous emission. In the new context of non-perfect but universal cloning following Bužek and Hillery, one is led to the question of how well cloning via stimulated emission could work. Could it be optimal? We will see that the answer is yes.

Let us first recall the basic facts about stimulated emission by considering the simplest possible model, namely a two–level system with levels e and g, coupled to a single mode a of the electro–magnetic field via the Hamiltonian

$$H = \gamma(|e\rangle\langle g|a + \text{h.c.}), \tag{1.16}$$

where γ is a coupling constant.

This is known as the Jaynes–Cummings model. The two–level system plays the role of inverted medium, i.e. it is supposed to be prepared in the excited state $|e\rangle$. Now consider the transition amplitudes for emission of a photon. For early times their size is determined by the matrix elements of the Hamiltonian between the respective states, i.e. by

$$\langle g|\langle n+1|H|e\rangle|n\rangle, \tag{1.17}$$

where we have assumed that n photons are already present in node a. From the commutation relation $[a, a^\dagger] = \mathbb{1}$ of the bosonic operator a it follows immediately that $\langle n+1|a^\dagger|n\rangle = \sqrt{n+1}$. Therefore this matrix element scales like $\sqrt{n+1}$. This implies that the two–level system is more likely to emit a photon in a given mode the more photons are already present in this mode. The photons that are already present stimulate the emission of new photons. This is of course the fundamental principle for the working of lasers. Spontaneous emission corresponds to the case where originally there are no photons in mode a, i.e. to the matrix element $\langle e|\langle 1|H|g\rangle|0\rangle$. Sometimes one says that in this case the emission is stimulated by the vacuum fluctuations.

Now imagine an inverted medium that can emit photons of different polarizations. Two modes, a_1 and a_2, are sufficient to describe polarization. For the sake of clarity and simplicity we are going to neglect the fact that in general there are different spatial modes. If the system is in some sense rotationally invariant and one sends in an a_1 photon, the emission of a_1 photons which is stimulated should be more likely than the emission of a_2 photons, which is only spontaneous. This implies that some kind of copying should occur. It is also clear that this copying procedure might not be perfect since spontaneous emission is going to happen, i.e. in general there will be unwanted a_2 photons. In fact, the presence of this spontaneous emission is unavoidable if one demands universality of the copying, as will become clear below. Note that if our inverted medium is really rotationally invariant this should ensure the universality of the cloning procedure. In such a situation spontaneous emission into both polarization modes will be equally likely, but because of stimulation the amplitude for emission into the desired mode will be larger, roughly by a factor $\sqrt{n+1}$, where n is the number of incoming photons, as pointed out above.

We will start by looking at perhaps the simplest model system which has all the required properties. It turns out that this is already perfectly suited to achieve optimal universal cloning. The inverted medium that we will use as a cloning device consists of an ensemble of Lambda-atoms. These are three-level systems that have two degenerate ground states $|g_1\rangle$ and $|g_2\rangle$ and an excited level $|e\rangle$. The ground states are coupled to the excited state by two modes of the electromagnetic field, a_1 and a_2, respectively. These two modes define the Hilbert space of our qubit to be cloned, i.e. we want to clone general superposition states $(\alpha a_1^\dagger + \beta a_2^\dagger)|0,0\rangle = \alpha|1,0\rangle + \beta|0,1\rangle$. We

can think of a_1 and a_2 as being orthogonal polarizations of one photon with a specific frequency, but we do not have to restrict ourselves to such a specific example, in fact we can think about other systems and other degrees of freedom, as long as they are described by the same formalism, e.g. a_1 and a_2 could also refer to the center-of-mass motion (phonons) in an ion trap. In the interaction picture, after the usual dipole and rotating wave approximations, the interaction Hamiltonian between field and atoms has the following form:

$$\begin{aligned} \mathcal{H}_i &= \gamma \left(a_1 \sum_{k=1}^{N} |e^k\rangle\langle g_1^k| + a_2 \sum_{k=1}^{N} |e^k\rangle\langle g_2^k| \right) + h.c. \\ &= \gamma \left(a_1 \sum_{k=1}^{N} \sigma_{+,1}^k + a_2 \sum_{k=1}^{N} \sigma_{+,2}^k \right) + h.c. \end{aligned} \tag{1.18}$$

The index k refers to the k-th atom. Note that in (1.18) the atoms couple to only one single spatial mode of the electromagnetic field. In particular this means that spontaneous emission into all other modes is neglected. Situations where this is a good approximation can now be achieved in cavity QED [49]. We also assume that the coupling constant γ is the same for all atoms, which in a cavity QED setting means that they have to be in equivalent positions relative to the cavity mode. Trapping of atoms inside a cavity has recently been achieved [93]. Finally note that our Hamiltonian has no spatial dependence, which means that the effect of the field on the motion of the atoms is neglected, their spatial wave-function is assumed to be unchanged. This leads to the question what the spatial wave-function could be. The most fascinating possibility would probably be to imagine a Bose–Einstein condensate.

The Hamiltonian (1.18) is invariant under simultaneous unitary transformations of the vectors (a_1, a_2) and $(|g_1\rangle, |g_2\rangle)$ with the same matrix U. If one furthermore chooses an initial state of the atoms that has the same invariance, then the system behaves equivalently for all incoming photon polarizations, i.e. universal cloning is achieved. This can be seen in the following way. Consider an incident photon in a general superposition state $(\alpha a_1^\dagger + \beta a_2^\dagger)|0,0\rangle$. Together with the orthogonal one-photon state this defines a new basis in polarization space, which is connected to the original one by a unitary transformation. If the atomic states are now rewritten in the basis that is connected to the original one by the same unitary transformation, then under the above assumptions the interaction Hamiltonian and initial state of the atoms look exactly the same as in the original basis. The initial state where all atoms are excited to $|e\rangle$ has the required invariance: it is completely unaffected by the above-mentioned transformations.

We can therefore, without loss of generality, restrict ourselves to the cloning of

photons in mode a_1. We consider an initial state

$$|\Psi_{in}\rangle = \otimes_{k=1}^{N} |e^k\rangle \frac{(a_1^\dagger)^m}{\sqrt{m!}} |0,0\rangle, \tag{1.19}$$

i.e. we are starting with m photons of a given polarization, and we want to produce a certain (larger) number n of clones.

1.4.1 The simplest case

For illustrative purposes let us first consider the simplest case of one Lambda-atom and one photon polarized in direction 1:

$$|\Psi_{in}\rangle = |e\rangle a_1^\dagger |0,0\rangle = |e\rangle|1,0\rangle =: |\mathcal{F}_0\rangle \tag{1.20}$$

To study the time development, we expand the evolution operator $e^{-i\mathcal{H}t}$ into a Taylor series and determine the action of powers of $\mathcal{H}$ on the state $|\Psi_{in}\rangle$.

$$\begin{aligned}
\mathcal{H}|\Psi_{in}\rangle &= \gamma(|g_1\rangle a_1^\dagger|1,0\rangle + |g_2\rangle a_2^\dagger|1,0\rangle) \\
&= \gamma\sqrt{3}\frac{(\sqrt{2}|g_1\rangle|2,0\rangle + |g_2\rangle|1,1\rangle)}{\sqrt{3}} =: \gamma\sqrt{3}|\mathcal{F}_1\rangle \\
\mathcal{H}^2|\Psi_{in}\rangle &= \gamma^2(|e\rangle a_1\sqrt{2}|2,0\rangle + |e\rangle a_2|1,1\rangle) = 3\gamma^2|e\rangle|1,0\rangle = 3\gamma^2|\mathcal{F}_0\rangle \\
&\dots
\end{aligned} \tag{1.21}$$

The result is

$$\begin{aligned}
e^{-i\mathcal{H}t}|\Psi_{in}\rangle &= \cos(\gamma\sqrt{3}t)|e\rangle|1,0\rangle - i\sin(\gamma\sqrt{3}t)(\sqrt{\frac{2}{3}}|g_1\rangle|2,0\rangle + \sqrt{\frac{1}{3}}|g_2\rangle|1,1\rangle) \\
&= \cos(\gamma\sqrt{3}t)|\mathcal{F}_0\rangle - i\sin(\gamma\sqrt{3}t)|\mathcal{F}_1\rangle
\end{aligned} \tag{1.22}$$

$|\mathcal{F}_0\rangle$ and $|\mathcal{F}_1\rangle$ denote the states of the system atom-photons that lie in the subspace with 1 and 2 photons respectively. $|\mathcal{F}_0\rangle$ is in the subspace where no cloning has taken place and $|\mathcal{F}_1\rangle$ in the one where one additional photon has been emitted, so that the two photons can now be viewed as clones with a certain fidelity. This way of labeling the states will turn out to be convenient below. The probability that the system acts as a cloner is $p(1) = \sin^2(\gamma\sqrt{3}t)$. The fidelity F_1 of the cloning procedure can be defined as the relative frequency of photons in the correct polarization mode in the final state $|\mathcal{F}_1\rangle$ (cf. Sec. 1.6). One finds

$$F_1 = \frac{2}{3}\cdot 1 + \frac{1}{3}\cdot\frac{1}{2} = \frac{5}{6}, \tag{1.23}$$

which is exactly the optimal fidelity for a 1-to-2 cloner, cf. Sec. 1.2. Actually, the state

$$|\mathcal{F}_1\rangle = \sqrt{\frac{2}{3}}|2,0\rangle|g_1\rangle + \sqrt{\frac{1}{3}}|1,1\rangle|g_2\rangle \tag{1.24}$$

is exactly equivalent to the three-qubit state

$$\sqrt{\frac{2}{3}}|11\rangle|\downarrow\rangle + \sqrt{\frac{1}{3}}\left(\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)\right)|\uparrow\rangle \tag{1.25}$$

produced by the Bužek-Hillery cloner. The equivalence is established, if the photonic states in Eq. (1.24) are identified with the corresponding *symmetrized* two-qubit states (both photons in mode 1 means both qubits in state $|1\rangle$, one photon in each mode means one qubit in state $|1\rangle$, one in state $|0\rangle$) in Eq. (1.25), while the atomic states $|g_1\rangle$ and $|g_2\rangle$ are identified with the states $|\downarrow\rangle$ and $|\uparrow\rangle$ of the ancillary qubit. This is another way of proving the optimality of Eq. (1.24). Note that in our case the *universality* follows directly from the symmetry of initial state and Hamiltonian, as explained above. In the following we show that a similar equivalence holds between our cloning scheme and the Gisin-Massar cloners in the completely general case (arbitrary numbers of photons and atoms).

1.4.2 Equivalence to coupled harmonic oscillators

We now turn to the discussion of the general case, i.e. we consider the initial state (1.19). We are going to show the equivalence of our system defined by (1.18) and (1.19) to a system of coupled harmonic oscillators. First note that both the initial state (1.19) and the Hamiltonian (1.18) are invariant under permutations of the atoms, which implies that the state vector of the system will always be completely symmetric. Furthermore the Hamiltonian (1.18) can be rewritten as

$$\mathcal{H} = \gamma\left(a_1 J_{+,1} + a_2 J_{+,2}\right) + h.c. \tag{1.26}$$

in terms of "total angular momentum" operators

$$J_{+,r} = \sum_{k=1}^{N} \sigma_{+,r}^{k} = \sum_{k=1}^{N} |e^k\rangle\langle g_r^k| \qquad (r = 1, 2), \tag{1.27}$$

By the above considerations one is led to use a Schwinger type representation [79] for the angular momentum operators:

$$J_{+,r} = b_r c^\dagger \qquad (r = 1, 2), \tag{1.28}$$

where $c^\dagger$ is a harmonic oscillator operator creating "e" type excitations, while b_1 destroys "g_1" excitations. Note that $J_{+,1}$ and $J_{+,2}$ share the operator $c^\dagger$ because

both ground levels g_1 and g_2 are connected to the same upper level e by the Hamiltonian (1.18), and correspondingly for the Hermitian conjugates. In terms of these operators, (1.18) acquires the form

$$\mathcal{H}_{osc} = \gamma(a_1 b_1 + a_2 b_2)c^\dagger + h.c., \tag{1.29}$$

while the initial state (1.19) is now given by

$$|\psi_i\rangle = \frac{(a_1^\dagger)^m}{\sqrt{m!}} \frac{(c^\dagger)^N}{\sqrt{N!}} |0\rangle = |m_{a1}, 0_{a2}, 0_{b1}, 0_{b2}, N_c\rangle \equiv |m, 0, 0, 0, N\rangle. \tag{1.30}$$

Actually, for reasons that will become apparent below, it is slightly more convenient for our purposes to use the following Hamiltonian instead of (1.29):

$$\mathcal{H} = \gamma(a_1 b_2 - a_2 b_1)c^\dagger + h.c., \tag{1.31}$$

which can be obtained from (1.29) by a simple unitary transformation in mode b, corresponding to a simple redefinition of the atomic states in (1.18). This is the Hamiltonian that is going to be used in the rest of this paper. The invariance properties of (1.31) are linked to those of (1.18) or equivalently (1.29) discussed above: (1.31) is invariant under simultaneous identical SU(2) transformations in modes a and b (because the determinant of such a transformation is equal to unity), while a phase transformation in either mode can be absorbed into γ. This ensures the universality of the cloning procedure.

We are now dealing with five harmonic oscillator modes defined by the operators $c, b_1, b_2, a_1,$ and a_2. Action of (1.31) on (1.30) generates Fock basis states of the general form

$$|(m+j)_{a1}, i_{a2}, i_{b1}, j_{b2}, (N-i-j)_c\rangle = |m+j, i\rangle_{photons} |i, j, N-i-j\rangle_{atoms}. \tag{1.32}$$

Remember that a_1 is now coupled to b_2 etc. Expressed in terms of individual atoms, $|i, j, N-i-j\rangle_{atoms}$ is the completely symmetrized state with i atoms in level g_1, j atoms in level g_2, and $N-i-j$ atoms in level e. The correctness of (1.28) can be checked by explicit application of left hand side and right hand side to such a general state, written in terms of the individual atoms and in terms of harmonic oscillator eigenstates respectively.

As noted above, the action of the Hamiltonian (1.18) on the initial state (1.19) only generates completely symmetric states of the atomic system. These states have the general form

$$\binom{N}{i,j}^{-1/2} \sum_\alpha |g_1^{\alpha_1}, g_1^{\alpha_2}, \ldots, g_1^{\alpha_i}, g_2^{\alpha_{i+1}}, \ldots, g_2^{\alpha_{i+j}}, e^{\alpha_{i+j+1}}, \ldots, e^{\alpha_N}\rangle$$
$$=: |i, j, N-i-j\rangle_{atoms} \tag{1.33}$$

where the sum is over all arrangements α of the $N-i-j$ levels $|e\rangle$, the i levels $|g_1\rangle$, and the j levels $|g_2\rangle$ on the N atoms, and $\binom{N}{i,j} = \frac{N!}{i!j!(N-i-j)!}$ is the multinomial coefficient giving the number of such arrangements.

Now study the action of a typical term in the Hamiltonian (1.18) on the system whose state we will write as
$|i,j,N-i-j\rangle_{atoms} \otimes |m+i,j\rangle_{photons}$:

$$\begin{aligned}
& (\sum_{k=1}^{N} |g_1^k\rangle\langle e^k|) a_1^\dagger |i,j,N-i-j\rangle_{atoms} \otimes |m+i,j\rangle_{photons} \\
= & \sum_{k=1}^{N} |g_1^k\rangle\langle e^k| \sqrt{\frac{i!j!(N-i-j)!}{N!}} \sum_\alpha |g_1^{\alpha_1},\ldots,g_1^{\alpha_i},g_2^{\alpha_{i+1}},\ldots,e^{\alpha_N}\rangle \\
& \otimes a_1^\dagger |m+i,j\rangle_{field} \\
= & (i+1)\sqrt{\frac{i!j!(N-i-j)!}{N!}} \sum_\alpha |g_1^{\alpha_1},\ldots,g_1^{\alpha_i},g_1^{\alpha_{i+1}},g_2^{\alpha_{i+2}},\ldots,e^{\alpha_N}\rangle \\
& \otimes a_1^\dagger |m+i,j\rangle_{field} \\
= & \sqrt{i+1}\sqrt{N-i-j}\sqrt{\frac{(i+1)!j!(N-i-j-1)!}{N!}} \\
& \sum_\alpha |g_1^{\alpha_1},\ldots,g_1^{\alpha_i},g_1^{\alpha_{i+1}},g_2^{\alpha_{i+2}},\ldots,e^{\alpha_N}\rangle \otimes a_1^\dagger |m+i,j\rangle_{field} \\
= & \sqrt{i+1}\sqrt{N-i-j}|i+1,j,N-i-j-1\rangle_{atoms} \otimes a_1^\dagger |m+i,j\rangle_{field} \quad (1.34)
\end{aligned}$$

Here the factor $(i+1)$ arises from the number of different configurations that a given arrangement α can be reached by. This shows that this term acts exactly like a term $a_1^\dagger b_1^\dagger c$. Similar calculations can be made for the other terms in the Hamiltonian. Together, they justify the Schwinger representation (1.28). Note that the use of the Schwinger representation is only convenient because the initial state of the atomic system in (1.19) is completely symmetric under permutation of the atoms.

Studying the Hamiltonian in the form (1.31) instead of (1.18) is helpful in several respects. The number of atoms N that is explicit in the Hamiltonian (1.18) now appears only as a part of the initial conditions of our system, which makes it easy to treat the general case of N atoms in one go. We will do this in the next subsection.

The Hamiltonian (1.31) can also be seen as a Hamiltonian for down-conversion with a quantized pump-mode described by the operator c, while a_r and b_r are the signal and idler modes respectively, where r labels the polarization degree of freedom. Usually in parametric down–conversion the operator c of (1.31) is replaced by a c–number. This corresponds to the limit of a classical pump field. These remarks lead to an experimental realization of optimal quantum cloning via stimulated emission

which we will discuss in Sec. 1.5.

In passing we note that the above dynamical equivalence generalizes to atoms with more than 2 ground-states $|g_n\rangle$ that are coupled each to a different degree of freedom of photons a_n. By similar arguments a system of N identical atoms with r ground states $\{|g_1\rangle, \ldots, |g_r\rangle\}$ governed by a Hamiltonian

$$\mathcal{H}^r = \gamma \sum_{k=1}^{N} \sum_{n=1}^{r} |e^k\rangle\langle g_n^k| a_n + h.c. \tag{1.35}$$

is equivalent to a system of $r+1$ coupled harmonic oscillators with lowering operators c and $b_1, \ldots, b_r$ governed by the interaction Hamiltonian

$$\mathcal{H}_{osc}^r = \gamma \sum_{n=1}^{r} c b_n^\dagger a_n^\dagger + h.c. \tag{1.36}$$

1.4.3 Cloning of m photons with N Lambda-atoms: Proof of optimality

We are now going to show that the system defined by (1.30) and (1.31) indeed realizes optimal cloning for arbitrary N and m. The idea of the proof is the following. After evolution in time the system that started with a certain photon number m will be in a superposition of states with different total photon numbers, where total means counting photons in mode a_1 and a_2, i.e. both "good" and "bad" copies. We will show that the general form of the state vector after a time interval t is

$$|\Psi(t)\rangle = e^{-i\mathcal{H}t}|\Psi_{in}\rangle = \sum_{l=0}^{N} f_l(t)|\mathcal{F}_l\rangle, \tag{1.37}$$

where l denotes the number of *additional* photons that have been emitted and

$$|\mathcal{F}_l\rangle := \binom{m+l+1}{l}^{-\frac{1}{2}} \sum_{i=0}^{l} (-1)^i \sqrt{\binom{m+l-i}{m}} |(m+l-i)_{a1}, i_{a2}, i_{b1}, (l-i)_{b2}, (N-l)_c\rangle. \tag{1.38}$$

Note that the number of photons can never become smaller than m since all the atoms start out in the excited state. $|\mathcal{F}_l\rangle$ is a normalized state of the system with $m+l$ photons in total. To see that $|\mathcal{F}_l\rangle$ is properly normalized note that $\sum_{i=0}^{l} \binom{m+i}{m} = \binom{m+l+1}{l}$.

The states $|\mathcal{F}_l\rangle$ are formally identical to the states obtained in [19], which have been shown to realize optimal universal cloning and the optimal universal NOT simultaneously. The ideal universal NOT is an operation that produces the orthogonal

complement of an arbitrary qubit. Like perfect cloning, it is prohibited by quantum mechanics. The transformation in [19] links universal cloning and universal NOT (anti-cloning): the ancilla qubits of the cloning transformation are the anti-clones. In our case, the clones are the photons in the a-modes and the anti-clones are the atoms in the b-modes (atomic ground states g_1 and g_2). From the Hamiltonian (1.31) and (1.38) it is clear that for every "good" emitted photon-clone (in mode a_1) there is an excitation in mode b_2 which corresponds to an anti-clone (atomic ground state $|g_2\rangle$). The only difference to the states in [19] is the presence of the fifth harmonic oscillator mode c, describing the "e" type excitations, which counts the total number of clones that have been produced (equal to the number of atoms having gone to one of the ground states) and doesn't affect any of the conclusions.

A distinguishing feature of our cloner is that the output state (1.38) is a superposition of states with different total numbers of clones. Cloning with a certain fixed number of produced copies can be realized by measuring the number of atoms in the excited state $|e\rangle$ (corresponding to mode c) and post-selection.

To prove that the system is indeed always in a superposition of the states $|\mathcal{F}_l\rangle$ as in Eq. (1.38) we use induction: The initial state of the system is $|\Psi_{in}\rangle = |\mathcal{F}_0\rangle$. Now we will show that if $|\Phi\rangle$ is a superposition of states $|\mathcal{F}_l\rangle$ then $\mathcal{H}|\Phi\rangle$ is so, too. Then, since $|\Psi(t)\rangle = e^{-i\mathcal{H}t}|\Psi_{in}\rangle = \sum_p \frac{(-i\mathcal{H}t)^p}{p!}|\Psi_{in}\rangle$ this implies that $|\Psi(t)\rangle$ will be a superposition of $|\mathcal{F}_l\rangle$. Explicit calculation shows that

$$\begin{aligned}
\mathcal{H}|\mathcal{F}_l\rangle &= \gamma(\sqrt{(l+1)(N-l)(m+l+2)}|\mathcal{F}_{l+1}\rangle \\
&\quad +\sqrt{l(N-l+1)(m+l+1)}|\mathcal{F}_{l-1}\rangle) \quad 1 \le l < N \\
\mathcal{H}|\mathcal{F}_0\rangle &= \gamma\sqrt{N(m+2)}|\mathcal{F}_1\rangle \\
\mathcal{H}|\mathcal{F}_N\rangle &= \gamma\sqrt{N(m+N+1)}|\mathcal{F}_{N-1}\rangle
\end{aligned} \tag{1.39}$$

which completes the proof.

Note that the form of the coefficients $f_l(t)$ didn't play any role in our proof. Actually, the f_l are in general hard to determine exactly. Solutions have been found in limiting cases. For the limit of a classical pump field (c replaced by a c–number), the solution can be found by standard methods and is given in Sec. 1.5 in the context of a proposed experimental realization of quantum cloning. The solution in the case of large incoming photon numbers ($m \gg N$) can be obtained in the following way.

For that case, the recursion (1.39) becomes

$$\begin{aligned}
\mathcal{H}|\mathcal{F}_l\rangle &= \gamma\sqrt{m}(\sqrt{(l+1)(N-l)}|\mathcal{F}_{l+1}\rangle + \sqrt{l(N-l+1)}|\mathcal{F}_{l-1}\rangle) \quad 1 \le l < N \\
\mathcal{H}|\mathcal{F}_0\rangle &= \gamma\sqrt{m}\sqrt{N}|\mathcal{F}_1\rangle \\
\mathcal{H}|\mathcal{F}_N\rangle &= \gamma\sqrt{m}\sqrt{N}|\mathcal{F}_{N-1}\rangle
\end{aligned} \tag{1.40}$$

It is possible to diagonalize the "transfer" matrix A acting on the vector $(f_0, \ldots, f_N)$ that corresponds to the action of $\mathcal{H}$ on $|\Psi\rangle = \sum_{l=0}^{N} f_l |\mathcal{F}_l\rangle$:

$$A_{l,l+1} = \gamma\sqrt{m}\sqrt{(l+1)(N-l)} = A_{l+1,l}. \tag{1.41}$$

This allows to exponentiate A and to determine the final state of the system after a time t:

$$|\Psi(t)\rangle = \sum_{l=0}^{N}(-i)^l \sqrt{\binom{N}{l}} \cos^{N-l}(\gamma\sqrt{m}t)\sin^l(\gamma\sqrt{m}t)|\mathcal{F}_l\rangle \tag{1.42}$$

Differentiating (1.42) and using (1.40) one can show that this state fulfills Schrödinger's equation with the correct initial condition.

In this big-m-limit the probability to observe the system as an $m \to m + l$ cloner (i.e. the probability that l additional photons are emitted) is

$$p(l) = \binom{N}{l} \cos^{2(N-l)}(\gamma\sqrt{m}t)\sin^{2l}(\gamma\sqrt{m}t) \tag{1.43}$$

This is a binomial distribution with a probability $\sin^2(\gamma\sqrt{m}t)$ for each atom to emit a photon. Setting $N = 1$ or comparison with Eq. (1.22) shows that this is identical to the probability for the case of only one atom in the case of large m. This means that in this limit each atom interacts independently with the electromagnetic field, because the effect of the other atoms on the field is negligible. In the short-time limit $p(l) = O(t^{2l})$. Furthermore the expected average number of "clones" $N_c = \sum_{l=0}^{N} l p(l) = N \sin^2(\gamma\sqrt{m}t)$ oscillates with an m-dependent frequency.

Let us pause here for a moment and summarize what we have found. Our system consisting of an ensemble of Lambda-atoms in the excited state is indeed equivalent to a superposition of optimal cloning machines a la Bužek-Hillery or Gisin-Massar, producing various numbers of clones. The atoms play the double role of photon source and of ancilla, the atomic ground states can be identified with the ancilla states in the qubit cloners. As for the corresponding qubit cloners, those ancillary atoms can also be seen as the output of a universal NOT gate. On the other hand, the atoms that end up in the excited state provide information about the number of clones that has actually been produced. This can be used to realize cloning and anti–cloning with a fixed number of output clones by post–selection.

1.4.4 The equivalence between pairs of V–atoms and Lambda–atoms

In this section we present an alternative (but similar) way of realizing optimal universal cloning that uses entangled pairs of V-atoms instead of Lambda atoms. We

prove optimality by showing that the system can be exactly mapped onto the system with Lambda atoms that we discussed above.

The two degenerate upper levels of each V-atom, $|e_1\rangle$ and $|e_2\rangle$, are coupled to the ground state $|g\rangle$ via the two orthogonal modes a_1 and a_2 respectively. The Hamiltonian describing the interaction of atom and field is:

$$\begin{aligned}\mathcal{H}_V &= \gamma\left(a_1^\dagger \sum_{k=1}^{N} |g^k\rangle\langle e_1^k| + a_2^\dagger \sum_{k=1}^{N} |g^k\rangle\langle e_2^k|\right) + \text{h.c.} \\ &= \gamma\left(a_1^\dagger \sum_{k=1}^{N} \sigma_{-,1}^k + a_2^\dagger \sum_{k=1}^{N} \sigma_{-,2}^k\right) + \text{h.c.} \end{aligned} \tag{1.44}$$

It arises from similar assumptions as (1.18). In contrast to before we now choose an entangled state of the atoms as the initial state. This is motivated by the fact that the initial atomic state has to be a *singlet* under polarization transformations in order for our cloning device to be again universal.

Let us first examine the simplest case of two entangled V-atoms, A and B, and one incoming photon. The initial state of the system is

$$|\Psi_{in}\rangle = \frac{1}{\sqrt{2}}(|e_1^A e_2^B\rangle - |e_2^A e_1^B\rangle) \otimes |1,0\rangle \tag{1.45}$$

Developing the time evolution operator $e^{-i\mathcal{H}_V t}$ into a power series, one finds easily:

$$\begin{aligned} e^{-i\mathcal{H}_V t}|\Psi_{in}\rangle &= \cos(\gamma\sqrt{3}t)\tfrac{|e_1^A e_2^B\rangle - |e_2^A e_1^B\rangle}{\sqrt{2}}|1,0\rangle \\ -i\sin(\gamma\sqrt{3}t)&\left(\sqrt{\tfrac{2}{3}}\tfrac{|g^A e_2^B\rangle - |e_2^A g^B\rangle}{\sqrt{2}}|2,0\rangle + \sqrt{\tfrac{1}{3}}\tfrac{|e_1^A g^B\rangle - |g^A e_1^B\rangle}{\sqrt{2}}|1,1\rangle\right) \end{aligned} \tag{1.46}$$

With the substitution

$$\begin{aligned} \frac{|e_1^A e_2^B\rangle - |e_2^A e_1^B\rangle}{\sqrt{2}} &\longrightarrow |\tilde{e}\rangle \\ \frac{|g^A e_2^B\rangle - |e_2^A g^B\rangle}{\sqrt{2}} &\longrightarrow |\tilde{g_1}\rangle \\ \frac{|e_1^A g^B\rangle - |g^A e_1^B\rangle}{\sqrt{2}} &\longrightarrow |\tilde{g_2}\rangle \end{aligned} \tag{1.47}$$

the state (1.46) has exactly the same form as the corresponding state (1.22) for one Lambda-atom, which implies that it also implements optimal universal $1 \to 2$ cloning.

Actually, the correspondence goes much further. Consider an initial atomic state consisting of N pairs of V-atoms, where each pair is in a singlet state:

$$|\psi_i\rangle = \otimes_{k=1}^{N} |\tilde{e}^k\rangle \tag{1.48}$$

with $|\tilde{e}\rangle$ as defined in (1.47).

It is easy to see that the action of the Hamiltonian (1.44) on each pair only generates one of the three antisymmetric atomic states in Eq. (1.47). Because of the invariance of the Hamiltonian under permutations, and in particular under the exchange of two atoms belonging to the same pair, transitions between states with different symmetry properties are impossible. In fact, with the identification (1.47) the Hamiltonian (1.44) has exactly the same form as the Hamiltonian for Lambda-atoms (1.18). The analysis made previously for Lambda atoms now goes through unchanged and we obtain the same cloning properties of a system of pairwise entangled V-atoms as we had before for Lambda-atoms, i.e. we have found another way of realizing optimal universal cloning. Although this scheme would without doubt be more difficult to realize experimentally, we believe that the underlying equivalence between the two systems is interesting and may be useful in other contexts as well.

1.4.5 Single V–Atoms are sub-optimal cloners

From the results of the previous sections, one might be tempted to conclude that the fulfillment of the symmetry requirements discussed above already implies optimality of the cloning procedure. Here we show that this is not the case by studying an explicit example of universal but suboptimal cloning via stimulated emission.

We are again considering an ensemble of V–atoms where each atom is initially in the mixed state

$$\rho_i = \frac{1}{2}(|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|), \tag{1.49}$$

which is invariant under the same unitary transformations. The invariance of both Hamiltonian and initial state together ensure the universality of the cloning procedure. Therefore it is sufficient to analyze the performance of the cloner for one arbitrary incoming one-photon state; we choose $|\psi_i\rangle = a_1^\dagger|0\rangle$.

We have performed numerical computations for systems of a few (up to $N = 6$) atoms. From (1.44), the time development operator $U = e^{-iHt}$ for the whole atoms-photons system was calculated. Use was made of the fact that N_1 and N_2, which denote the sum of the number of photons plus the number of excited atoms for mode 1 and 2 respectively, are independently conserved quantities. Therefore the whole Hilbert space is decomposable into invariant subspaces, i.e. H and U are block-diagonal.

The final state of the procedure is an entangled state of the atom-photon system that has components with various numbers of photons, where the maximum possible total number is $N + 1$ (if all atoms have emitted their photons). The probability to find k "right" and l "wrong" photons in the final state, denoted by $p(k, l)$, was

calculated for all possible values of k and l and for different values of γt, and from it the overall average "fidelity"

$$f_{clones}(t) = \sum_{k+l\geq 2} p'(k,l;t) \left(\frac{k}{k+l}\right) \tag{1.50}$$

was determined. This is the average of the relative frequency of photons with the correct polarization in the final state. Note that in (1.50) the average is performed only over those cases where there are at least two photons in the final state, i.e. where at least one clone has been produced. $p'(k,l) = p(k,l)/(1-p(1,0)-p(0,1))$ is used in order to have proper normalization. Note that $p(0,0)$ is always zero.

That average fidelity for our cloning procedure was compared to the average fidelity that would be achieved by an ensemble of optimal cloners producing the same distribution of numbers of photons, i.e. to

$$f_{opt}(t) = \sum_{n=2}^{N+1} p'(n;t) \left(\frac{2n+1}{3n}\right), \tag{1.51}$$

where $p'(n) = \sum_{k+l=n} p'(k,l)$. We also made a comparison to the case, where, in addition to the incoming photon, photons are just created randomly, i.e. to the fidelity

$$f_{rand}(t) = \sum_{n=2}^{N+1} p'(n;t) \left(\frac{n+1}{2n}\right). \tag{1.52}$$

Fig. 1.1 shows clearly that the fidelity of our cloning procedure approaches the optimum fidelity for early times. One can also see that for longer interaction times f_{clones} departs from f_{opt} and even becomes lower than f_{rand}. This behavior, which may seem surprising, is due to the fact that for longer times absorption of photons by atoms that have already emitted once and gone to the ground-state becomes important. Note that absorption of "right" photons is favored if there are more such photons present. In particular, also the incoming "right" photon can be absorbed by an atom that has emitted a "wrong" photon before, resulting in departure from optimality for later times. The superiority of f_{rand} in that regime is understandable because in our idealized random cloner the incoming photon is always left intact.

Our computations show that the system goes through many emission-reabsorption cycles, though without exhibiting a simple periodicity. As a consequence, over long times f_{clones} oscillates taking values above and below f_{rand}, sometimes approaching f_{opt} again.

Fig. 1.2, which also illustrates the above-mentioned cyclic behavior of our system, shows the time dependence of the mean number of photons and of the mean number

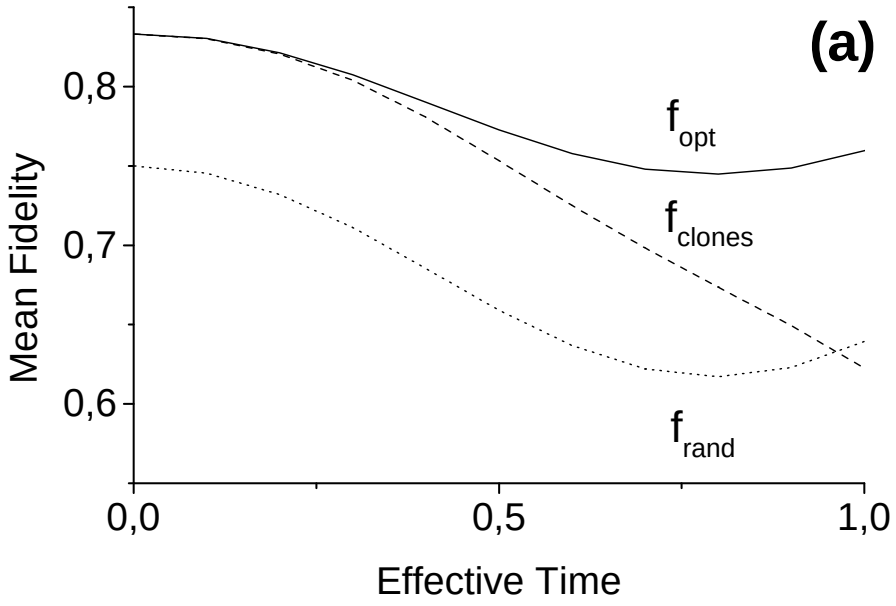


Figure 1.1: Dependence on time, measured in units of γt, of f_{opt}, f_{clones}, and f_{rand}, which are the optimum possible fidelity, the fidelity achieved by our V–atom cloning procedure, and the fidelity achieved by random photon production respectively, as defined in Eqs. (1.51,1.50,1.52), for the case of $N = 6$ atoms. It is evident that optimal cloning is achieved in the short-time limit. The behavior for lower atom numbers is the same.

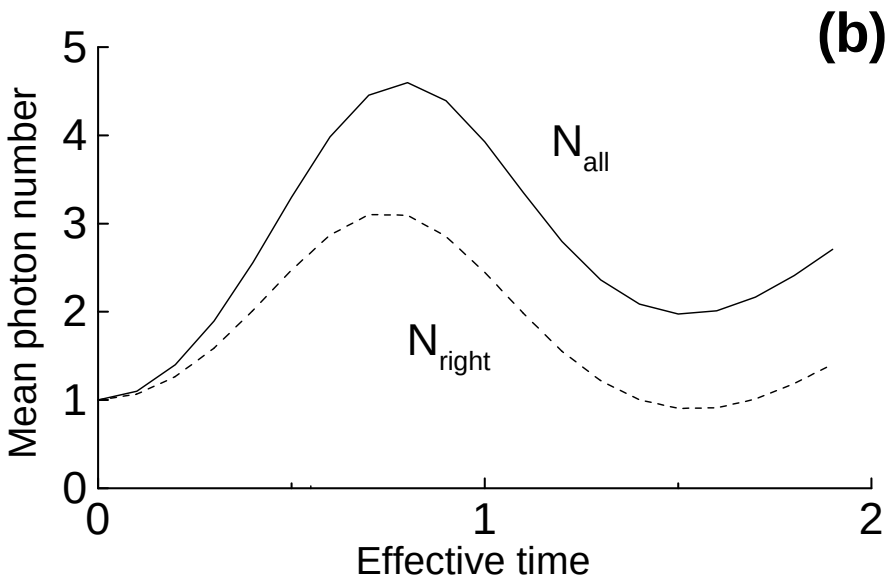


Figure 1.2: Time dependence of the mean number of all photons N_{all} and of the mean number of "right" photons (i.e. of the same polarization as the incoming photon) N_{right} for the case $N = 6$.

of photons of the correct polarization. For short times, which is the interesting regime from the point of view of cloning, the probability for every individual atom to have already emitted its photon is low. Therefore, in order to produce a reasonable average number of clones in this regime, a large number of atoms is necessary.

Our results show that symmetry alone is not sufficient to achieve optimal cloning. But note that even in this case optimality is approached for short interaction times.

1.5 Experimental realization

Here we propose a concrete experimental realization of the ideas discussed in the previous sections. The scheme for quantum cloning that we want to present is based on stimulated parametric down-conversion (PDC). We will show that optimal cloning can be realized with present technology. In PDC a strong light beam is sent through a crystal. There is a certain (very low) probability for a photon from the beam to decay into two photons such that energy and crystal momentum are conserved. In type-II PDC the two photons that are created have different polarization. They are denoted as signal and idler.

Fig. 1.3 shows the setup that we have in mind. We consider pulsed type-II frequency-

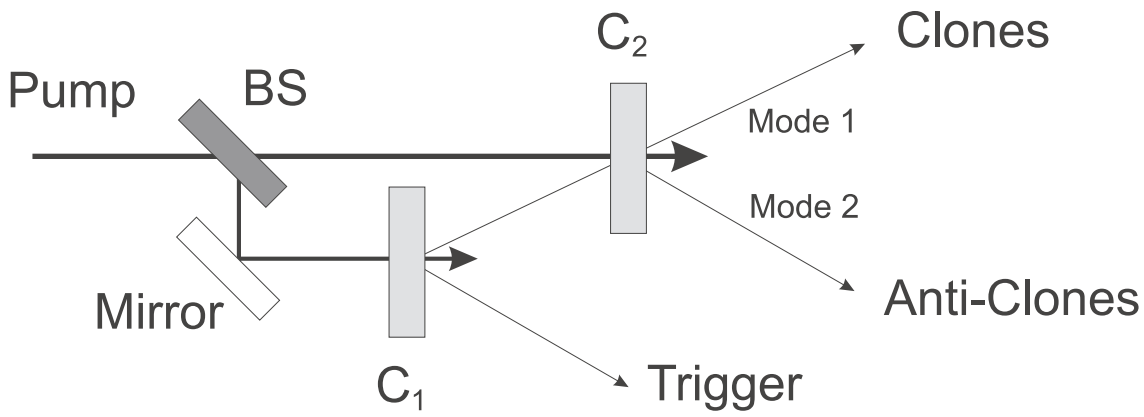


Figure 1.3: Setup for optimal cloning by parametric down-conversion [61, 35, 67]. The pump-pulse is split at the beam splitter BS. One part of the pump pulse hits the first crystal C_1, where photon pairs are created with a certain rate. One photon from each pair can be used as a trigger. The other photon is the system to be cloned. This photon is directed towards the second crystal C_2, where it stimulates emission of photons of the same polarization along the same direction. The path lengths have to be adjusted in such a way that the DC-photon and the second part of the pump pulse reach C_2 simultaneously. The photons in mode 1 are optimal clones of the incoming photon, and the photons in mode 2 are the output of an optimal universal NOT-gate. It is interesting to note that in this scheme one is actually cloning a photon that is part of an entangled pair.

degenerate PDC. It is possible to choose two conjugate directions for the signal and idler beams such that photon pairs that are created along these two directions are entangled in polarization [59]. We consider the quasi-collinear case (i.e. the two directions almost coincide), so that the transverse motion of the photons in the crystal is not important.

For stimulated emission to work optimally, there has to be maximum overlap of the amplitudes of the incoming photon and of all the photons that are produced in the second crystal. This can be achieved by using a pulsed scheme together with filtering of the photons before detection [95]. The pump pulse can be seen as an active volume that moves through the crystal. If the photons are filtered so much that the smallest possible size of the wavepackets detected is substantially bigger than the pump pulse, then there is maximum overlap between different pairs created in the same pulse. Of course, filtering limits the achievable count rates. Moreover the group velocities of pump pulse, signal (V) and idler (H) photons are not all identical. This leads to separations (of the order of a few hundred fs per millimeter in BBO), which have to be kept small compared to the size of the DC-photon wave packets. There is a trade-off between filtering and crystal length, i.e. one can choose narrower filters in order to be able to use a longer crystal (which leads to longer interaction times).

If the above-mentioned conditions are fulfilled, then a single spatial mode (i.e. one mode for the signal and one for the idler photons) approximation can be used. The PDC process can then be described in the limit of a large classical pump pulse, in the interaction picture, by the Hamiltonian

$$H = \gamma(a_{V1}^\dagger a_{H2}^\dagger - a_{H1}^\dagger a_{V2}^\dagger) + h.c., \tag{1.53}$$

where $a_{V1}^\dagger$ is the creation operator for a photon with polarization V propagating along direction 1 etc. The coupling constant and the intensity of the classical pump pulse are contained in γ. As discussed above this Hamiltonian corresponds to the limit of the Λ–atom Hamiltonian (1.18) for a coherent state of mode c.

The Hamiltonian H is invariant under simultaneous general $SU(2)$ transformations of the polarization vectors $(a_V^\dagger, a_H^\dagger)$ for modes 1 and 2, while a phase transformation will only change the phase of γ. This makes our cloner universal, i.e. its performance is polarization independent. Therefore it is again sufficient to analyze the "cloning" process in one basis.

The time development operator e^{-iHt} clearly factorizes into a $V1 - H2$ and an $H1 - V2$ part. Consider cloning starting from N identical photons in the initial state $|\psi_i\rangle = \frac{(a_{V1}^\dagger)^N}{\sqrt{N!}}|0\rangle$ Making use of the disentangling theorem [89, 31] one finds

that (cf. [35])

$$|\psi_f\rangle = e^{-iHt}|\psi_i\rangle = K \quad \sum_{k=0}^{\infty}(-i\Gamma)^k\sqrt{\binom{k+N}{N}}|k+N\rangle_{V1}|k\rangle_{H2} \times \quad \sum_{l=0}^{\infty}(i\Gamma)^l|l\rangle_{H1}|l\rangle_{V2} \tag{1.54}$$

where $\Gamma = \tanh\gamma t$ and K is a normalizing factor. This reproduces the results of the previous section c.f. Eq. (1.37,1.38). We see, that in this limiting case it is possible to determine the coefficients $f_l(t)$ of Eq. (1.37) explicitly.

The component of this state which has a fixed number M of photons in mode 1, is proportional to

$$\sum_{l=0}^{M-N}(-1)^l\sqrt{\binom{M-l}{N}}|M-l\rangle_{V1}|l\rangle_{H1}|l\rangle_{V2}|M-N-l\rangle_{H2}. \tag{1.55}$$

This is identical to the state produced by the unitary transformation written down in [19] which can be seen as a special version of the Gisin-Massar cloners [44] that implements optimal universal cloning and the optimal universal NOT-gate at the same time. The M photons in mode 1 are the clones, while the $M-N$ photons in mode 2, which act as ancillas for the cloning, are the output of the universal NOT-gate, the "anti-clones".

This means that the setup of Fig. 1.3 works as an ensemble of optimal universal cloning (and universal NOT) machines, producing different numbers of clones and anti-clones with certain probabilities. Note that each of the modes can be used as a trigger for the other one and therefore cloning or anti-cloning with a fixed number of output-systems can be realized by post-selection.

We have shown a method of realizing optimal quantum cloning machines. We emphasize that this scheme should be experimentally feasible with current technology. In our group, pair production probabilities of the order of $4\cdot10^{-3}$ have been achieved with a 76 MHz pulsed laser system (UV-power about 0,3 W) and a 1 mm BBO crystal, for 5 nm filter bandwidth. Past experiments show that good overlap of photons originating from different pairs is achieved under these conditions. With detection efficiencies around 10 percent, this leads to a rate of two-pair detections of the order of one per a few seconds.

Let us note that experiments in the spirit of the present proposal are currently under way in at least two laboratories [35, 14]. First results were reported in [36].

1.6 Cloning of photons versus cloning of qubits

In this section we are going to discuss the physical differences that exist in spite of the formal equivalence proven above between our photon cloners based on stimulated emission and the qubit cloners as usually considered [21, 44]. In particular, we will show that the claim that optimal cloning is realized by our devices is justified in spite of these differences.

In most of the previous work cloning was discussed in terms of quantum networks. In general, the situation considered in these papers is the following: one has a certain number of qubits that are localized in different positions, which makes them perfectly distinguishable. At the beginning, some of those qubits are the systems to be cloned, the others play the role of ancillas. After the cloning procedure, which consists of several joint operations on the qubits that can be expressed in terms of quantum gates, some of the qubits are the clones, the rest are ancillas, which for a specific form of the optimal cloning transformation can also be seen as outputs of the universal NOT operation. As a consequence of localization, it is possible to address individual clones.

In our stimulated emission cloners, the situation is different. All input systems (photons) are in the same spatial mode (called mode a), and, even more importantly, all clones are produced into that mode. Note that this is completely unavoidable if stimulated emission is to be used. One can say that this is the price one has to pay for the great conceptual simplicity of the cloning procedure itself.

However, having all clones in the same spatial mode is not necessarily an important disadvantage. For example, if perfect cloning of that kind were possible, one could still determine the polarization of the original photon to arbitrary precision by performing measurements on the clones. This would still make superluminal communication possible. It may be interesting to note that in the paper that started the whole discussion about quantum cloning, Herbert [50] considered cloning via stimulated emission and therefore necessarily into a single spatial mode.

If one wants to distribute the clones to different locations, this can for example be achieved using an array of beam splitters. However, this does not lead to a situation where one can be sure to have exactly one photon in each mode. If one wants to have at most one photon in each mode, the array has to have many more output modes than there are photons.

Another distinguishing feature of our cloners compared to the usual qubit cloners is the fact that the same procedure is used to produce different numbers of clones. While in the qubit case the network to be used depends on the number of desired clones, in our case the final state is a superposition of states with different numbers

of clones. Of course, the average number of clones produced depends on the number of atoms present in the system and the interaction time. As discussed in Sec. 1.4 cloning with a fixed number of output clones can be achieved by post–selection based on a measurement of the number of excited atoms in the final state.

The formal equivalence between the qubit cloners and our one-mode cloners can arise because the output state produced by the optimal qubit cloners is completely symmetric under the exchange of clones [21, 44]. Because of the bosonic nature of the photons there is a one-to-one-mapping between completely symmetric qubit states and photonic states. Note that asymmetric cloning could not be realized by the presented method. For a completely symmetric qubit state the two concepts of relative frequency of qubits in the "correct" basis state and of single-particle fidelity are equivalent. This can be seen in the following way. Let $|\psi\rangle$ denote the state that is to be copied. Then the usual definition of the (single–particle) cloning fidelity is

$$F = \langle\psi|\rho_{red}|\psi\rangle, \tag{1.56}$$

where ρ_{red} is the reduced density matrix of one of the clones, say the first one, i.e.

$$\rho_{red} = \mathrm{Tr}_{2,3,\ldots,N}\left[\,\rho\,\right] \tag{1.57}$$

Then F can also be expressed as

$$F = \mathrm{Tr}\left[\,\rho\,|\psi\rangle\langle\psi|_1 \otimes I_2 \otimes \ldots \otimes I_N\right]. \tag{1.58}$$

On the other hand, the relative frequency of qubits in the state $|\psi\rangle$ can be written as

$$\frac{1}{N}\mathrm{Tr}\left[\,\rho\,\left(|\psi\rangle\langle\psi|_1 \otimes I_2 \otimes \ldots \otimes I_N + I_1 \otimes |\psi\rangle\langle\psi|_2 \otimes \ldots \otimes I_N + \ldots + I_1 \otimes \ldots \otimes |\psi\rangle\langle\psi|_N\right)\right]. \tag{1.59}$$

If ρ is invariant under exchange of any two clones, it is obvious that (1.59) is equal to (1.58), i.e. for symmetric cloners the two concepts are completely equivalent. This justifies our definition of fidelity via the relative frequency in the case of photon cloning (cf. Sec. 1.4).

Let us finally address the issue of optimality in the context of stimulated emission cloners. In this paper we have shown the formal equivalence of our scheme and the optimal schemes for qubit cloning. As a consequence, the fidelity of the clones saturates the bounds derived for the cloning of qubits. However, it is not entirely obvious that the bounds derived for the situation of distinct well-localized qubits also apply to our situation. Could one maybe achieve even higher fidelity in our one-mode case? The following argument shows that the bounds indeed apply in our situation as well, i.e. that photon cloning is not allowed to be better than qubit cloning.

Let us assume that we had a single-mode cloning machine that clones photons with a better fidelity than given by the bounds for qubits. Consequently, the relative frequency of "correct" photons has to exceed the bound for at least one value of the final total photon number M. This is obvious if M has been fixed by post-selection. Otherwise the fidelity has to be defined as the average of the relative frequencies over all final total photon numbers. This average can only exceed the bound for qubits if the bound is violated for at least one particular value M of the final photon number.

As a consequence, we have a universal map from the N-photon Hilbert space to the M-photon Hilbert space that achieves a relative frequency of correct photons in the final state that is higher than the qubit bound. But the existence of such a map is equivalent to the existence of a universal map from the totally symmetric N-qubit space to the totally symmetric M-qubit space with a single-particle fidelity equal to the relative frequency. The existence of the latter map is excluded by the theorems on cloning of qubits [91]. This justifies our claim that the schemes presented in the previous sections realize *optimal* cloning of photons.

1.7 Why are our cloners optimal?

On the previous pages we have shown that optimal cloning can indeed be realized by stimulated emission. The states produced by our simple quantum optical model systems consisting of two photonic modes and an ensemble of Lambda systems are exactly the same as those derived by Bužek and Hillery and Gisin and Massar. But why is that so? An element of wonder seems to remain. It follows from symmetry considerations that our systems should act as universal cloners, but a priori optimality was not necessarily to be expected. We have seen in subsection 1.4.5 that not all systems which have the required symmetries also lead to optimal universal cloning.

The reasons behind the optimality of our cloning procedure can be understood by remembering our construction of the optimal universal cloning transformations in section 1.3, where we saw that the output of the optimal cloner is given by the projection of the state

$$\psi_1 \dots \psi_N \, S_{N+1,M+1} \dots S_{M,2M-N} \tag{1.60}$$

onto the completely symmetric subspace of the first M qubits, where S can be replaced by the maximally entangled state $\frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$.

On the other hand, one can show that the final states of our stimulated emission cloners are always linear combinations of states of the following form:

$$(a_1^\dagger b_1^\dagger + a_2^\dagger b_2^\dagger)^{M-N} {a_1^\dagger}^N |0\rangle, \tag{1.61}$$

where the c modes are disregarded because they just count the number of photons, as explained above. Here an initial state with N photons in mode 1 was assumed and we have already chosen the exponent $M-N$ in such a way as to facilitate the comparison to (1.60). That the final state can indeed be expressed in terms of (1.61) can be seen by noting that all terms generated during the time evolution will be of the form

$$H^k {a_1^\dagger}^N |0\rangle \tag{1.62}$$

for some k, where

$$H = \gamma c(a_1^\dagger b_1^\dagger + a_2^\dagger b_2^\dagger) + h.c. \tag{1.63}$$

Furthermore the commutator

$$[a_1 b_1 + a_2 b_2, a_1^\dagger b_1^\dagger + a_2^\dagger b_2^\dagger] = N_a + N_b + \mathbb{1}, \tag{1.64}$$

where $N_a = a_1^\dagger a_1 + a_2^\dagger a_2$ etc. This means that we can get rid of all annihilation operators by commuting them to the right.

The similarity between the two expressions (1.60) and (1.61) is obvious. The ${a_1^\dagger}^N$ in (1.61) corresponds to the N instances of ψ in (1.60), while the $(a_1^\dagger b_1^\dagger + a_2^\dagger b_2^\dagger)^{M-N}$ corresponds to the $M-N$ instances of S. The projection onto the completely symmetric subspace is built in automatically in (1.61) through the commutation properties of the bosonic operators. This intuitive explanation can be checked by explicitly evaluating the completely symmetrized component of (1.60), and (1.61). One verifies that the density matrices of the a–modes are indeed identical in both cases. Thus we have finally understood the formal equivalence between our stimulated emission cloning procedures and the classical optimal cloners.

1.8 Conclusions and Outlook

One may feel that with the remarks in the previous section the work is really completed. Cloning via stimulated emission has been shown to be realizable, and the formal reasons for its optimality are understood. Furthermore, we proposed a concrete experimental realization which should lead to results in the near future. The initial intuition probably shared by many physicists that a gain medium is something like a cloner was thus shown to be entirely correct. Our study demonstrates the intimate connection between the apparently deeply quantum-field-theoretical concept of stimulated emission and the quantum-information concept of cloning. It is the author's hope that there may be more things to learn about quantum field theory by looking at it from a quantum- information point of view.

Another moral of the present work refers to the technology of quantum information. It reminds us that at least for specific tasks there may sometimes be more natural

and therefore possibly also more practical implementations than quantum computing networks. Concerning possible practical applications of the present work it is worth mentioning that the optimal universal cloner constitutes the optimal eavesdropping method in some protocols for quantum cryptography [6], so conceivably a future Eve could rely on stimulated emission. We have discussed a possible implementation using parametric down-conversion in some detail. Other implementations might be possible, most notably based on combining cavity QED and Bose-Einstein condensation. It should be mentioned that parametric down-conversion-like Hamiltonians can be realized for BECs, which makes them natural candidates for the implementation of the cloning of atomic states.

2 The No–Signaling Condition

2.1 Introduction

The special theory of relativity is one of the cornerstones of our present scientific world-view. One of its most important features is the fact that there is a maximum velocity for signals, i.e. for anything that carries information, identical to the velocity of light in vacuum. Within the special theory of relativity, superluminal communication would immediately lead to all kinds of causal paradoxes, e.g. one would be able to influence one's own past.

Another cornerstone of our present understanding of the world is quantum physics. Quantum physics seems to have "nonlocal" characteristics due to quantum entanglement. Most importantly, it is not compatible with local hidden variables, as shown by the violation of Bell's inequalities [7], which has been experimentally confirmed in several experiments [5, 90].

It is very remarkable that in spite of its non-local features, quantum mechanics is compatible with the special theory of relativity, if it is assumed that operators referring to space-like separated regions commute. In particular, one cannot exploit quantum-mechanical entanglement between two space-like separated parties for communication of classical messages faster than light [40].

This peaceful coexistence between quantum physics and special relativity has led physicists to ask whether the principle of the impossibility of superluminal communication, which we will refer to as the "no-signaling condition", could be used as an axiom in deriving basic features of quantum mechanics. Here we show that it is indeed possible. If the usual *kinematical* characteristics of quantum mechanics are assumed, then its *dynamical* rules can be derived from the no-signaling assumption. By quantum kinematics we mean the following: the states of our systems are described by vectors in a Hilbert space, and the usual rules for the results of measurements apply, including the projection postulate. However, no a priori assumption is made about the time evolution of the system.

Our result is then, more precisely, that under the stated conditions the dynamics of our system has to be described by *completely positive (CP) linear* [77] maps.

This is equivalent to saying that under the given assumptions quantum mechanics is essentially the only option since according to the Kraus representation theorem [77], every CP map can be realized by a quantum-mechanical process, i.e. by a unitary (linear) evolution on a larger Hilbert space (while on the other hand any quantum process corresponds to a CP map). This rather surprising result is an extension of earlier work by N. Gisin [41].

In the following, we will first recall how superluminal communication is impossible in quantum mechanics in spite of the existence of entangled states, as a consequence of the linearity of quantum dynamics. Then we show that quantum kinematics and no-signaling imply quantum dynamics. The argument proceeds in two steps. Firstly, it is shown that the existence of entanglement, the projection postulate and the no-signaling constraint imply linearity. Secondly, complete positivity of the dynamics follows from the the existence of entanglement and linearity.

2.2 No–signaling in Quantum Mechanics

Consider two parties, denoted by Alice and Bob, who are space-like separated, which implies that all operations performed by Alice commute with all operations performed by Bob. (Throughout this work we will assume that in *this* sense locality is implemented in the quantum kinematics.) Can they use a shared entangled state ψ_{AB} in order to communicate in spite of their space-like separation?

The short answer is: no, because the situation on Bob's side will always be described by the same reduced density matrix, whatever Alice chooses to do. All the effects of her operations (described by linear maps) disappear when her system is traced over. This answer is correct, but not very detailed, and thus it may not be entirely convincing. In particular, a question that is frequently raised in this context is the following: Alice could choose to measure her system in two different bases and thus project Bob's system into different pure states depending on the basis she chose and her measurement result. Since it is possible to distinguish two different states in quantum mechanics, at least with some probability, shouldn't it be possible for Bob to infer her choice of basis, at least in some percentage of the cases (which would be dramatic enough)?

Of course, the answer is no again, for the following reason. In order to gain information about which basis Alice chose to measure, Bob can only perform some (generalized) measurement on his system. Then he has to compare the conditional probabilities for this result to occur, for the case that Alice measured in the first or in the second basis. But these conditional probabilities will always be exactly the same for both possibilities. This can be seen as a consequence of the linearity

of quantum mechanics: Suppose that Alice's first choice projects Bob's system into states ψ_i with probabilities p_i and her second choice projects it into states ϕ_μ with probabilities q_μ. Bob can calculate the probability for his obtained result in every one of the states, and then weight these probabilities with the probability to have this specific state. But because of the linearity of any operation that Bob can perform on his states during his generalized measurement procedure, his final result will only depend on the density matrix of the probabilistic mixtures, which is the same in both cases, because they were generated from the same entangled state. For an example how two such mixtures can become distinguishable through a non-linear (non-quantum-mechanical) evolution, see [42].

Let us note that this argument also implies the non-existence of a perfect cloner in quantum mechanics because such a machine would allow superluminal communication [50]. The impossibility of perfect cloning can also be shown directly from the linearity of quantum mechanics [92].

2.3 No-signaling and Linearity

In this section we show how quantum dynamics can be derived from quantum kinematics and the no-signaling condition. By quantum kinematics we mean that the usual Hilbert space–structure (including entanglement) and the projection postulate are assumed. The probabilities of the results of measurements are assumed to be determined by the density matrices of the systems in the usual way. Thus if we consider a subsystem of the whole Universe it will in general be in an entangled state with other parts of the Universe. In particular, it may also happen that a system denoted by A is entangled with another system B which is *space-like* separated with respect to A, such that their observable algebras do commute. This is where the no-signaling constraint comes into play. The dynamics of the systems has to be such that in spite of this entanglement no superluminal communication between A and B is possible.

Suppose that A and B together are in the entangled state $|\psi\rangle_{AB}$ with reduced density matrix ρ_A for system A. As a consequence of the projection postulate, by performing a measurement of his system the observer B also prepares a certain state in A. In particular, *every* probabilistic mixture of pure states corresponding to the density matrix ρ_A can be prepared via appropriate measurements on B (for a proof see Sec. (2.4) and Ref. [52]).

Consider two such probabilistic mixtures $\{P_{\psi_i}, x_i\}$ and $\{P_{\phi_j}, y_j\}$, where P_{ψ_k} is the

projector corresponding to the pure state $|\psi_k\rangle$ and x_k is its probability, such that

$$\sum_i x_i P_{\psi_i} = \sum_j y_j P_{\phi_j} = \rho_A. \tag{2.1}$$

According to the no-signaling principle there should be no way for the observer in A to distinguish these different probabilistic mixtures.

A general dynamical evolution in system A is of the form

$$g : P_\psi \to g(P_\psi) \tag{2.2}$$

where, most importantly, g is not necessarily linear. Furthermore, $g(P_\psi)$ does not have to be a pure state. Firstly, it could evolve into a mixed state. Secondly, if ψ evolves into a probabilistic mixture, then we define $g(P_\psi)$ to denote the corresponding density matrix. Under such dynamics the probabilistic mixture $\{P_{\psi_k}, x_k\}$ goes into another probabilistic mixture $\{g(P_{\psi_k}), x_k\}$. Therefore the two final density matrices after the action of g on two different probabilistic mixtures $\{P_{\psi_i}, x_i\}$ and $\{P_{\phi_j}, y_j\}$ are

$$\begin{aligned} \rho'_A\{P_{\psi_i}, x_i\} &= \sum_i x_i g(P_{\psi_i}) \\ \rho'_A\{P_{\phi_j}, y_j\} &= \sum_j y_j g(P_{\phi_j}) \end{aligned} \tag{2.3}$$

which *a priori* can be different. Let us recall that according to our assumptions the results of all measurements in A are determined by the reduced density matrix ρ'_A. This means that as a consequence of the no-signaling principle the density matrix ρ'_A at any later time has to be the same for all probabilistic mixtures corresponding to a given initial density matrix ρ_A. That is, it has to be a function of ρ_A only.

We can therefore write

$$\rho'_A = g(\rho_A) = g(\sum_i x_i P_{\psi_i}). \tag{2.4}$$

From the above it also follows that

$$\rho'_A = \sum_i x_i g(P_{\psi_i}), \tag{2.5}$$

therefore g satisfies the condition

$$g(\sum_i x_i P_{\psi_i}) = \sum_i x_i g(P_{\psi_i}), \tag{2.6}$$

which implies that the map g is *linear*. Let us stress that there are three crucial ingredients in our argument: the existence of entanglement, the projection postulate, and the no-signaling condition. Specifically, the projection postulate leads to

probabilistic mixtures and thus to the right-hand side of Eq. (2.6). On the other hand, the no-signaling condition tells us that the dynamics can depend only on the reduced density matrix, which leads to the left-hand side of Eq. (2.6). Positivity is necessary in order to ensure that $g(\rho_A)$ is again a valid density matrix, i.e. to ensure the positivity of all probabilities calculated from it.

As we have made no specific assumptions about the system A (apart from the fact that it can be entangled with some other system), this means that the dynamics of our theory has to be linear in general.

Let us now argue that linearity and positivity already imply *complete positivity* in the present framework. To see this, consider again two arbitrary subsystems A and B which may again be in an entangled state $|\psi\rangle_{AB}$. Now it is conceivable that system A is changed locally (i.e. the system evolves, is measured etc.), which is described by some operation g_A, while *nothing* happens in B. This formally corresponds to the operation $g_A \otimes \mathbb{1}_B$ on the whole system. Strictly speaking, we have made an additional assumption here, namely that the identity operation $\mathbb{1}_B$ on a subsystem is a valid physical operation.

The joint operation $g_A \otimes \mathbb{1}_B$ should take the density matrix of the composite system ρ_{AB} into another valid (i.e. positive) density matrix, whatever the dimension of the system B. But this is exactly the definition of *complete positivity* for the map g_A [77]. If g_A is positive but not CP, then by definition there is always some entangled state ρ_{AB} for which $g_A \otimes \mathbb{1}_B(\rho_{AB})$ is no longer a positive density matrix and thus leads to unphysical results such as negative probabilities. Let us recall that transposition of system A is an example for a positive but non-CP map.

In this way the existence of entangled states and the requirements of positivity and linearity actually force us to admit only completely positive dynamics. As mentioned already in the introduction, this is equivalent to saying that under the given assumptions quantum dynamics is essentially the *only* option since any CP map can be realized by a quantum mechanical process, and on the other hand, any quantum-mechanical process corresponds to a CP map.

Let us recall once again our starting assumptions: these were the existence of entanglement, the projection postulate, the no-signaling condition, and, strictly speaking, the assumption that the identity operation on a subsystem is a permitted dynamical evolution. Nonlinear modifications of quantum mechanics [34] have to give up at least one of these assumptions. For instance, if the dynamics is allowed to depend only on the reduced density matrix ρ_A, but in a nonlinear way, then it is clear that ρ_A cannot correspond to a probabilistic mixture of pure states. But ρ_A will correspond to such a mixture whenever the observer in B chooses to make an appropriate measurement, as long as we believe in the projection postulate. This implies that the projection postulate has to be modified in such a nonlinear theory. Another

example would be a theory where some entangled states are *a priori* excluded from the kinematics. In this case some non-CP maps might be permissible. An extreme example would be a theory without entanglement. Such a theory would of course be in conflict with experiments. An example for a linear, positive, but non-CP map consistent with the no-signaling condition is the transposition of the density matrix of the whole Universe. However in this case the identity operation on a subsystem is not an allowed dynamics.

2.4 Preparation of any mixture at a distance

Let us now show that any mixture corresponding to a given density matrix can be prepared at a distance from any entangled state with the correct reduced density matrix [41, 52]. Let us denote the system under consideration by A and the remote system by B. An immediate requirement on the state of the joint system $|\psi\rangle_{AB}$ in order to achieve this is that it needs to have the correct reduced density matrix ρ_A. Let us denote the eigenvector representation of ρ_A by

$$\sum_{k=1}^{r} \lambda_k |v_k\rangle\langle v_k|. \tag{2.7}$$

Then $|\psi\rangle_{AB}$ must have a Schmidt decomposition

$$|\psi\rangle_{AB} = \sum_{k=1}^{r} \sqrt{\lambda_k}\, |v_k\rangle|g_k\rangle, \tag{2.8}$$

where the $|g_k\rangle$ are orthonormal states of system B. We want to show that any decomposition of ρ_A as a mixture of pure states can be prepared from this state by operations on system B only. To this end, consider an arbitrary decomposition

$$\rho_A = \sum_{i=1}^{m} x_i |\psi_i\rangle\langle\psi_i|, \tag{2.9}$$

where in general $m > r$. Clearly this decomposition could be obtained from a state

$$|\phi\rangle_{AB} = \sum_{i=1}^{m} \sqrt{x_i}\, |\psi_i\rangle|\alpha_i\rangle, \tag{2.10}$$

with the $|\alpha_i\rangle$ being an orthonormal basis of a m–dimensional Hilbert space H_m. It seems that we now require a larger Hilbert space in location B in order to accommodate all the orthonormal $|\alpha_i\rangle$. But the state $|\phi\rangle_{AB}$ also has a Schmidt representation

$$|\phi\rangle_{AB} = \sum_{k=1}^{r} \sqrt{\lambda_k}\, |v_k\rangle|h_k\rangle, \tag{2.11}$$

which implies that $|\phi\rangle_{AB}$ and $|\psi\rangle_{AB}$ are connected by a unitary transformation on B alone:

$$|\phi\rangle_{AB} = \mathbb{1}_A \otimes U_B |\psi\rangle_{AB}, \tag{2.12}$$

so the dimension of the support of the reduced density matrix ρ_B is the same for both states. This means that $|\phi\rangle_{AB}$ can be rewritten as

$$|\phi\rangle_{AB} = P_S|\phi\rangle_{AB} = \sum_{i=1}^{m} \sqrt{x_i}\,|\psi_i\rangle|\tilde{\alpha}_i\rangle, \tag{2.13}$$

where $|\tilde{\alpha}_i\rangle = P_S|\alpha_i\rangle$ and P_S is the projector onto the support of ρ_B, i.e. the span of the vectors $|h_k\rangle$. The vectors $|\tilde{\alpha}_i\rangle$ live in this r–dimensional span. Let us recall that $|\phi\rangle_{AB}$ can be generated from $|\psi\rangle_{AB}$ by the local unitary U_B.

The states $|\psi_i\rangle$ can now be prepared with the help of the POVM

$$\sum_{i=1}^{m} |\tilde{\alpha}_i\rangle\langle\tilde{\alpha}_i| = 1 \tag{2.14}$$

on system B. Note that the $|\tilde{\alpha}_i\rangle$ form a POVM by construction: They can be extended to the orthonormal basis $\{|\alpha_i\rangle\}$ on the larger Hilbert space H_m. The proof of our above statement uses exactly this extension. For a general POVM $\sum P_l = 1$ on system B the result i prepares the state $\rho_A^i = \mathrm{Tr}_B(P_i)_B\rho_{AB}$ in system A. Therefore the state prepared by the result corresponding to $|\tilde{\alpha}_i\rangle$ is equal to $\rho_A^i = \mathrm{Tr}_B|\tilde{\alpha}_i\rangle\langle\tilde{\alpha}_i||\phi\rangle_{AB}\langle\phi|_{AB}$, which can be rewritten, formally extending the Hilbert space dimension for system B, as

$$\mathrm{Tr}_B P_S|\alpha_i\rangle\langle\alpha_i|P_S|\phi\rangle_{AB}\langle\phi| = \mathrm{Tr}_B|\alpha_i\rangle\langle\alpha_i||\phi\rangle_{AB}\langle\phi| = x_i|\psi_i\rangle\langle\psi_i|. \tag{2.15}$$

This completes the proof of our above statements [77]. Any decomposition of ρ_A in a mixture of pure states can be obtained from the state $|\psi\rangle_{AB}$.

2.5 Bounds on cloning from positivity and linearity

The motivation for the present section is partially historical. In [43] the no-signaling condition in the form of [41], which was just positivity and linearity, was used to derive a bound on the simplest cloner. According to our above argumentation, *all* of quantum dynamics can be derived from the no-signaling condition, so the bounds from no-signaling are identical to the quantum-mechanical bounds, which in the case of cloning are known [18, 91]. Nevertheless, the method of [43] is quite convenient for deriving bounds on universal quantum machines [84]. We illustrate it here for the

case of $1 \to N$ cloning, thus providing an alternative (maybe more direct) derivation of the corresponding bounds. We will first recall Gisin's treatment of the $1 \to 2$ cloner, then we generalize to the $1 \to N$ case.

Let the initial state of the input qubit be denoted by $\rho_0 = \frac{1}{2}(1+\vec{\sigma}\cdot\vec{m}) = |+\vec{m}\rangle\langle+\vec{m}|$, where $\vec{\sigma}\cdot\vec{m}|+\vec{m}\rangle = +|\vec{m}\rangle$. The output two-qubit density matrix is denoted by $\rho(\vec{m})$. From the discussion of the preceding section we know that $\rho(\vec{m})$ has to be a linear function of $\vec{m}$.

The output density matrix $\rho(\vec{m})$ is further constrained by the requirement of universality, which takes the form

$$\rho(U\vec{m}) = U \otimes U\rho(\vec{m})U^\dagger \otimes U^\dagger \tag{2.16}$$

for all unitary operators U. This implies that $\rho(\vec{m})$ depends only on $\vec{m}$ and on no other privileged direction. Thus, if $\rho(\vec{m})$ is written in the basis of matrices

$$\mathbb{1} \otimes \mathbb{1}, \mathbb{1} \otimes \sigma_i, \sigma_i \otimes \mathbb{1}, \sigma_i \otimes \sigma_k, \tag{2.17}$$

the coefficients can only depend on the components m_i of $\vec{m}$ and on the invariant tensors δ_{ij} and ϵ_{ijk}.

It is thus necessarily of the form:

$$\rho(\vec{m}) = \frac{1}{4}\left(\mathbb{1} \otimes \mathbb{1} + \eta_1 \vec{m}\vec{\sigma} \otimes \mathbb{1} + \eta_2 \mathbb{1} \otimes \vec{m}\vec{\sigma} + t\ \vec{\sigma} \otimes \vec{\sigma} + t_{xy}\vec{m}(\vec{\sigma} \wedge \vec{\sigma})\right) \tag{2.18}$$

where $\eta_1, \eta_2, t, t_{xy}$ are real parameters. In order for $\rho(\vec{m})$ to be a physical density matrix, its eigenvalues have to be non-negative. A simple calculation shows that this implies

$$\begin{aligned} 1 + t \pm (\eta_1 + \eta_2) &\geq 0 \\ 1 - t \pm \sqrt{4t^2 + 4t_{xy}^2 + (\eta_1 - \eta_2)^2} &\geq 0 \end{aligned} \tag{2.19}$$

In the case of symmetric cloning, the task is to optimize the fidelity $F = Tr(\rho(\vec{m})\ P_{\vec{m}} \otimes \mathbb{1})$, where $P_{\vec{m}} = |+\vec{m}\rangle\langle+\vec{m}|$, assuming $\eta_1 = \eta_2 \equiv \eta$. A simple calculation leads to the optimal values $t_{xy} = 0, t = 1/3, \eta = 2/3$, for which $F = \frac{5}{6}$. Note that this also optimizes $Tr(\rho(\vec{m})\ P_{\vec{m}} \otimes P_{\vec{m}}) = \frac{2}{3}$. These are exactly the bounds that are valid in quantum mechanics, cf. Sec. 1.2.

Now we show, that the above result can be generalized to the case of $1 \to N$ cloning. Firstly, from any non-covariant and non-permutation invariant cloning transformation that produces a number of copies that are scaled versions of the input qubit with identical scaling factors, by averaging over unitary transformations and permutations one can get a covariant and permutation invariant transformation

without affecting the quality of the copies [57]. This means that in deriving bounds we can restrict our attention to covariant and permutation invariant output density matrices.

Secondly, as shown above, it follows from no–signaling that ρ_{out} has to be linear in the Bloch vector of the input qubit.

Thus for building the output density matrix we are only left with m_i and δ_{jk} as possible coefficients in the Pauli matrix representation, where m_i can only occur linearly. The invariant tensor ϵ_{ijk} is excluded by the requirement of permutation invariance. Possible terms are:

$$\begin{gathered}
\mathbb{1} \otimes \mathbb{1} \otimes ... \otimes \mathbb{1} \\
\sigma_k \otimes \sigma_k \otimes \mathbb{1} \otimes ... \otimes \mathbb{1} + perm. \\
\sigma_k \otimes \sigma_k \otimes \sigma_l \otimes \sigma_l \otimes \mathbb{1} \otimes ... \otimes \mathbb{1} + perm. \\
.... \\
\vec{\sigma} \cdot \vec{m} \otimes \mathbb{1} \otimes ... \otimes \mathbb{1} + perm. \\
\vec{\sigma} \cdot \vec{m} \otimes \sigma_k \otimes \sigma_k \otimes \mathbb{1} \otimes ... \otimes \mathbb{1} + perm. \\
.... \qquad\qquad ,
\end{gathered} \tag{2.20}$$

where summation over repeated indices is understood. Because of universality we can choose $\vec{m}$ e.g. along the z-axis. Then one can convince oneself that all the above terms can be generated by products of

$$J_z = \frac{1}{2}(\sigma_z \otimes \mathbb{1} \otimes ... \otimes \mathbb{1} + ...) \tag{2.21}$$

and

$$\vec{J}^2 = J_k J_k, \tag{2.22}$$

the z-component of the total angular momentum and its square respectively. J_z can occur only linearly, while for $\vec{J}^2$ higher powers are possible: one has to distinguish the cases of N even and N odd. For $N = 2k$ $(\vec{J}^2)^n$ can go up to $n = k$ and $J_z(\vec{J}^2)^n$ up to $n = k - 1$, while for $N = 2k + 1$ $(\vec{J}^2)^n$ can go up to $n = k$ and $J_z(\vec{J}^2)^n$ up to $n = k$ as well. Higher powers are linearly dependent. We will denote the maximum possible values as n_{max} and n'_{max} in the following. Note that $n_{max} + n'_{max} = N - 1$.

Therefore the most general output density matrix can be written in the following way:

$$\rho_{out} = \beta_0 \mathbb{1} + \sum_{n=1}^{n_{max}} \beta_n (\vec{J}^2)^n + \alpha_0 J_z + \sum_{n=1}^{n'_{max}} \alpha_n J_z (\vec{J}^2)^n, \tag{2.23}$$

where $\mathbb{1}$ now denotes the unit matrix in the N-particle Hilbert space. Our task is to find coefficients α_i and β_i such that the scaling factor of an individual clone is

maximal. The constraints are given by the requirements of positivity and normalization: all eigenvalues of ρ_{out} have to be positive, and its trace has to be equal to unity.

In order to express the positivity constraint one has to diagonalize the matrix, but in the present formulation this is trivial. The eigenvectors are just the angular momentum eigenvectors $|j, m, \gamma_j\rangle$, where γ_j runs over the different irreducible representations for a given j, i.e. $\gamma_j = 1, ..., d_j$, where d_j denotes the number of irr. reps for j. This means that the eigenvalues are given by

$$\lambda(j, m) = \beta_0 + \sum_{n=1}^{n_{max}} \beta_n (j(j+1))^n + \alpha_0 m + \sum_{n=1}^{n'_{max}} \alpha_n m (j(j+1))^n. \tag{2.24}$$

(The eigenvalues do not depend on γ_j.) Positivity implies that

$$\lambda(j, m) \geq 0 \qquad \forall j, m. \tag{2.25}$$

The normalization constraint is

$$\mathrm{Tr}\rho_{out} = \beta_0 2^N + \sum_{n=1}^{n_{max}} \beta_n \mathrm{Tr}(\vec{J}^2)^n = 1, \tag{2.26}$$

because the trace of the terms with J_z is zero. This can be expressed as

$$\beta_0 2^N + \sum_{n=1}^{n_{max}} \sum_{j=j_{min}}^{N/2} \beta_n d_j (2j+1)(j(j+1))^n = 1. \tag{2.27}$$

For $N = 2k$ $j_{min} = 0$, while for $N = 2k+1$ $j_{min} = 1/2$.

We still have to determine the scaling factor of the individual clones, which is the quantity that we want to maximize. This requires calculation of the one-particle reduced density matrix, which in our case must have the form

$$\mathrm{Tr}_{N-1}\rho_{out} = \frac{1}{2}(\mathbb{1} + s\sigma_z). \tag{2.28}$$

We want to maximize the coefficient of σ_z. The terms in ρ_{out} leading to a σ_z are

$$\alpha_0 \mathrm{Tr}_{N-1} J_z + \sum_{n=1}^{n'_{max}} \alpha_n \mathrm{Tr}_{N-1} J_z (\vec{J}^2)^n. \tag{2.29}$$

The scaling factor s is obtained by multiplication with σ_z and tracing over the remaining particle. Using

$$\mathrm{Tr}_1 \left(\sigma_z \mathrm{Tr}_{N-1} J_z (\vec{J}^2)^n \right) = \frac{2}{N} \mathrm{Tr}_N (J_z)^2 (\vec{J}^2)^n, \tag{2.30}$$

$$\mathrm{Tr}_N(J_z)^2(\vec{J}^2)^n = \sum_{j=j_{min}}^{N/2} d_j(j(j+1))^n \sum_{m=-j}^{j} m^2, \tag{2.31}$$

and

$$\sum_{m=-j}^{j} m^2 = \frac{1}{3}j(j+1)(2j+1) \tag{2.32}$$

one obtains

$$s = \alpha_0 2^{N-1} + \frac{2}{3N}\sum_{n=1}^{n'_{max}} \alpha_n \sum_{j=j_{min}}^{N/2} d_j j(j+1)(2j+1)(j(j+1))^n. \tag{2.33}$$

By the structure of Eqs. (2.24),(2.27), and (2.33) one is led to make the substitution

$$\begin{aligned} a_j &= \sum_{n=1}^{n'_{max}} \alpha_n (j(j+1))^n \qquad j = j_{min}, ..., N/2 \\ b_j &= \sum_{n=1}^{n_{max}} \beta_n (j(j+1))^n \qquad j = j_{min}, ..., N/2. \end{aligned} \tag{2.34}$$

Note that a_0 and b_0, which arise for even N because $j_{min} = 0$, are identically zero. Note also that a priori this does not seem to be a good change of variables for the optimization because in general the a_j and b_j are not all linearly independent, as one can see by counting their number and comparing to the number of α_n and β_n. We will discuss this problem in detail below when we present the real change of variables made, the above substitution is only an intermediate step.

In these variables the optimization problem has the following form:

$$\beta_0 + b_j + (\alpha_0 + a_j)m \geq 0 \quad \forall j, m$$

$$\begin{aligned} \beta_0 2^N + \sum_{j=j_{min}}^{N/2} b_j d_j (2j+1) &= 1 \\ \alpha_0 2^{N-1} + \tfrac{2}{3N} \sum_{j=j_{min}}^{N/2} a_j d_j j(j+1)(2j+1) &= \text{Max.!} \end{aligned} \tag{2.35}$$

Noting that

$$\sum_{j=j_{min}}^{N/2} d_j(2j+1) = 2^N, \tag{2.36}$$

because it is the number of dimensions of all irreducible representations, and that

$$\sum_{j=j_{min}}^{N/2} d_j(2j+1)j(j+1) = \mathrm{Tr}(\vec{J}^2) = 3N2^{N-2}, \tag{2.37}$$

as can be checked by direct calculation of the trace, one is led to make the further redefinition

$$\begin{aligned} A_j &= (\alpha_0 + a_j) d_j (2j+1) \qquad j = j_{min}, ..., N/2 \\ B_j &= (\beta_0 + b_j) d_j (2j+1) \qquad j = j_{min}, ..., N/2. \end{aligned} \tag{2.38}$$

In terms of the independent variables α_n and β_n this reads

$$\begin{aligned} A_j &= d_j(2j+1) \left(\alpha_0 + \sum_{n=1}^{n'_{max}} \alpha_n (j(j+1))^n \right) \qquad j = j_{min}, ..., N/2 \\ B_j &= d_j(2j+1) \left(\beta_0 + \sum_{n=1}^{n_{max}} \beta_n (j(j+1))^n \right) \qquad j = j_{min}, ..., N/2 \end{aligned} \tag{2.39}$$

Now we have to face the question whether this is a legal change of variables, i.e. whether the A_j and B_j are linearly independent. Let's first discuss the second line of (2.39). There are $n_{max}+1$ independent parameters on the right hand side, which is $k+1$ for $N = 2k$ and also $k+1$ for $N = 2k+1$ (see above). This is identical to the number of different possible values of j. This means that the number of B_j is the same as the number of β_n, the only question left is whether the matrix connecting the two sets of variables is invertible. This last point is easy to show. (It's determinant is a Van der Monde determinant.)

Turning to the first line of (2.39) we see that the number of independent parameters on the right hand side is $n'_{max}+1$, which is k for $N = 2k$ and $k+1$ for $N = 2k+1$. This means that there seems to be a problem for the case $N = 2k$, because one of the A_j is a linear combination of the others. Fortunately it turns out, as we will see below, that the variable A_0 does not play any role in the optimization, which allows us to disregard it. The other A_j for $j \neq 0$ can be shown to be linearly independent exactly as the B_j.

Having justified our change of variables, we can now study its consequences. It leads to the following set of conditions:

$$\begin{aligned} \lambda(j,m) = B_j + A_j m \geq 0 \qquad & \forall j, m \\ \sum_{j=j_{min}}^{N/2} B_j = 1 \qquad & \\ \tfrac{2}{3N} \sum_{j=j_{min}}^{N/2} A_j j(j+1) = \text{Max.!} \qquad & \end{aligned} \tag{2.40}$$

From the first and third line one sees that A_0 only enters multiplied by zero and therefore doesn't play any role, as mentioned above. From the first line it follows that the B_j have to be positive, and also that $j|A_j| \leq B_j$. From the third line it is clear that negative values of A_j are not helpful, therefore one obtains

$$jA_j \leq B_j \quad \forall j. \tag{2.41}$$

If all these inequalities are saturated, one gets

$$s = \frac{2}{3N} \sum_{j=j_{min}}^{N/2} B_j(j+1). \tag{2.42}$$

From the above it is clear that the maximum is obtained if the value of B_j for the largest possible j, i.e. for $j = N/2$, is equal to unity, with all other B_j equal to zero. This leads to

$$s_{max} = \frac{1}{3} + \frac{2}{3N}, \tag{2.43}$$

which is exactly the maximum possible scaling factor in quantum mechanics.

2.6 Conclusions

We find it quite remarkable that what we referred to as quantum kinematics, i.e. the Hilbert space structure and the projection postulate, together with the no-signaling condition already constrains the dynamics to be of the form that we know from quantum mechanics: linear and completely positive. Concerning the challenge to truly derive quantum mechanics from some fundamental principles, the present result is certainly just a small piece of the puzzle. However, besides providing some insight into the interrelations between different properties of the standard theory, this result also leads to a clear statement about possible non-linear modifications of quantum mechanics, namely that they have to give up at least one of the assumptions made in our derivation. Although the author has some sympathy for the program of studying non-linear extensions, at present he is not sure which assumption he would be most willing to give up.

3 A Simple Kochen-Specker Experiment

3.1 Introduction

Most predictions of quantum mechanics are of a statistical nature, with the theory making probabilistic predictions for individual events. The question whether one can go beyond quantum mechanics in this respect, i.e. whether there could be hidden variables determining the results of all individual measurements, has been answered to the negative for *local* hidden variables by Bell's theorem [7]. Locality means that in such theories the results of measurements in a certain space-time region are independent of what happens in a space-time region that is space-like separated, in particular independent of the settings of a distant measuring apparatus.

Bell's theorem refers to a situation where there are two particles and where the predictions of quantum mechanics are statistical. Furthermore, even definite (non-statistical) predictions of quantum mechanics are in conflict with a local realistic picture for systems of three particles or more [47, 46].

The Kochen–Specker (KS) theorem [85, 58, 8, 75] states that *non–contextual* hidden variable theories are incompatible with quantum mechanics. Non-contextuality (NC) means that the value for an observable predicted by such a theory does not depend on the experimental context, i.e. which other co-measurable observables are measured simultaneously.

To put the Kochen-Specker (KS) theorem in a proper context, let us briefly recall some basic facts about measurements in quantum mechanics. Let us first discuss sequential measurements. Consider the sequential measurement of two commuting observables A and B. Ideally the same values for A and B are found again and again for repeated measurements, provided that they are projective. A measurement of B does not seem to disturb the value of A. The quantum mechanical explanation for this phenomenon is that the first measurement of A and the first measurement of B project the system into a joint eigenstate or eigenspace of A and B in which it remains.

For sequential measurements of non-commuting observables the situation is radically different. For example, consider repeated measurements of σ_z and σ_x. When

measuring the sequence σ_z, σ_x, σ_z one may find σ_z equal to -1 in the third measurement after having found σ_z equal to $+1$ in the first one. This can be visualized with polarizers and photons. This shows that if there are hidden values for quantum mechanical observables they must necessarily be affected by the measurement. But originally, before a measurement was performed, there might still have been one precise value for every observable, which was then influenced by the first measurement performed on the system.

Let us now consider joint measurements of several observables. In quantum mechanics only joint measurements of commuting observables are meaningful, because only these have joint eigenstates onto which the measurement can project. One can ask the following question: Is it possible that for all observables there are hidden values that do no depend on which other observables are measured jointly? For sequential measurements we have already seen that measurements of non-commuting observables have an effect. But here the situation is more subtle. We are talking about a situation where some observable A could be measured jointly with B or with C and we ask whether there can be an underlying theory such that the value for A does not depend on whether B or C are measured jointly, and such that this holds for all A, B and C. The KS theorem states that there can be no hidden values of this kind: measurements of commuting observables also matter.

The KS theorem was an important station on the road leading to Bell's theorem. While one can argue that there is no very good justification for expecting non-contextuality, if A, B and C are all measured on a single particle, as in the original formulation of the KS theorem, this changes dramatically, if A on the one hand and B and C on the other hand can be measured on two particles in entirely different locations. One can say that Bell's discovery was that the KS result remains true in such a situation as well. For the hypothetical hidden values measurements of commuting observables matter, even if they commute because of space-like separation.

Let us briefly recall the setting of the original KS theorem. KS considered a single spin-one particle, the relevant observables are the squares of the spin components along arbitrary directions, denoted e.g. by S_x^2 for the direction x. These observables commute for orthogonal directions. They satisfy the constraint

$$S_x^2 + S_y^2 + S_z^2 = s(s+1) = 2 \tag{3.1}$$

for all orthogonal triplets of directions $\{x, y, z\}$.

The question of non-contextuality now poses itself in the following way: is it possible to assign values 0 and 1 to all directions such that the constraint (3.1) is fulfilled? Thus the question of the existence of non-contextual hidden values becomes a coloring problem on the sphere. The non-existence of such a coloring can be inferred from Gleason's theorem [45]. Kochen and Specker gave a direct proof by exhibiting

a finite set of directions (originally 117) that cannot be colored. Since then proofs that require only smaller numbers of directions have been found, see e.g. Ref. [74].

It is well known that Bell's theorem leads to possible experimental tests of local hidden variables by studying the violation of certain inequalities for correlation functions. While tests of local hidden variables can also be seen as tests of non-contextuality, as briefly explained above, so far there has not been an experiment based on the original form of the Kochen-Specker theorem. This would require testing that the constraint Eq. (3.1) is indeed fulfilled for all directions belonging to the Kochen-Specker set. But the message of the KS theorem is weaker than that of Bell's theorem: non-contextual hidden variables are a smaller class than local hidden variables. It should therefore be possible to find an experiment disproving non-contextuality that is considerably simpler than the usual tests of Bell's inequalities. Furthermore, from the theoretical point of view, the KS argument is quite elaborate. It should be possible to reach the same conclusions in a much simpler way. Note that the GHZ argument can already be seen as a much simplified KS theorem. In the present chapter we show that the above programme can be realized. We present a simple argument against non-contextual theories which involves just a few observables and leads to a simple experiment. The present work was inspired by the work of Cabello and García-Alcaine (CG) [24].

The experiment can be realized with single particles, using their path and spin degrees of freedom. It leads to a non-statistical test of non-contextuality versus quantum mechanics. In this respect it is similar to the GHZ argument against local realism.

If the experiment is realized with photons, the setup that we shall present only requires a source for single photons (such as parametric down-conversion) and passive optical elements. In the following, we first show how a very direct experimental test of non-contextuality can be found, then we discuss our operational realization.

3.2 A Simple Kochen–Specker Argument ...

Consider four binary observables Z_1, X_1, Z_2, and X_2. Let us denote the two possible results for each observable by ± 1. In a non-contextual hidden variable (NCHV) theory these observables have predetermined non-contextual values $+1$ or -1 for individual systems, denoted as $v(Z_1), v(Z_2), v(X_1)$, and $v(X_2)$. This means e.g. that for an individual system the result of a measurement of Z_1 will always be $v(Z_1)$ irrespective of which other compatible observables are measured simultaneously.

Now imagine an ensemble E of systems for which one always finds equal results for Z_1 and Z_2, and also for X_1 and X_2. (Clearly, in order for this statement to be

meaningful, Z_1 and Z_2, and X_1 and X_2 have to be co-measurable.) In the NCHV theory this means that

$$v(Z_1) = v(Z_2) \quad \text{and} \quad v(X_1) = v(X_2) \tag{3.2}$$

for each individual system of the ensemble. Then there are only two possibilities: either $v(Z_1) = v(X_2)$, which implies $v(X_1) = v(Z_2)$; or $v(Z_1) \neq v(X_2)$, which implies $v(X_1) \neq v(Z_2)$. We will see that this elementary logical deduction is already sufficient to establish a contradiction between NCHV theories and quantum mechanics.

To this end, let us express the above argument in a slightly different way. Eq. (3.2) can be written as

$$v(Z_1)v(Z_2) = v(X_1)v(X_2) = 1. \tag{3.3}$$

Multiplying by $v(X_2)v(Z_2)$ it immediately follows that

$$v(Z_1)v(X_2) = v(X_1)v(Z_2). \tag{3.4}$$

Let us now introduce the notion of product observables such as Z_1X_2. By definition, one way of measuring Z_1X_2 is to measure Z_1 and X_2 separately and multiply the results; in general, there are other ways. In particular, if another compatible observable (e.g. X_1Z_2, cf. below) is measured simultaneously, it will in general not be possible to obtain separate values for Z_1 and X_2. However, in a non-contextual theory, the result of a measurement of an observable must not depend on which other observables are measured simultaneously. Therefore the predetermined value $v(Z_1X_2)$, for example, in a NCHV theory has to follow the rule [24]

$$v(Z_1X_2) = v(Z_1)v(X_2). \tag{3.5}$$

In this new language, our above argumentation can be resumed in the following way:

$$v(Z_1Z_2) = v(X_1X_2) = 1 \Rightarrow v(Z_1X_2) = v(X_1Z_2) \tag{3.6}$$

i.e. if our systems have the property expressed in Eq. (3.2), then the two product observables Z_1X_2 and X_1Z_2 must always be equal in a NCHV theory. Note that in general this prediction of NCHV can only be tested if Z_1X_2 and X_1Z_2 are co-measurable.

It follows from the results of [24] that the prediction (3.6) leads to an observable contradiction with quantum mechanics. To see this, consider a system of two qubits and the observables [24]

$$Z_1 := \sigma_z^{(1)}, X_1 := \sigma_x^{(1)}, Z_2 := \sigma_z^{(2)}, X_2 := \sigma_x^{(2)}, \tag{3.7}$$

where $\sigma_z^{(1)}$ means the z-component of the "spin" of the first qubit etc. It is easy to check that this set of observables satisfies all the properties required above. In

particular, while Z_1 and X_1, and Z_2 and X_2, do not commute, the two product observables Z_1X_2 and X_1Z_2 do. Furthermore, the quantum-mechanical two-qubit state

$$\begin{aligned} |\psi_1\rangle &= \frac{1}{\sqrt{2}}(|+z\rangle|+z\rangle + |-z\rangle|-z\rangle) \\ &= \frac{1}{\sqrt{2}}(|+x\rangle|+x\rangle + |-x\rangle|-x\rangle) \end{aligned} \tag{3.8}$$

is a joint eigenstate of the commuting product observables Z_1Z_2 and X_1X_2 with both eigenvalues equal to $+1$. Therefore, on the one hand the ensemble described by this state possesses the property of the ensemble E discussed above (cf. (3.2)): the measured values of Z_1Z_2 and X_1X_2 are equal to $+1$ for every individual system. On the other hand, quantum mechanics predicts for the state $|\psi_1\rangle$, that the measured value of Z_1X_2 will always be opposite to the value of X_1Z_2. This can be seen by decomposing $|\psi_1\rangle$ in the basis of the joint eigenstates of the two commuting product observables Z_1X_2 and X_1Z_2:

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|\chi_{1,-1}\rangle + |\chi_{-1,1}\rangle), \tag{3.9}$$

with

$$\begin{aligned} |\chi_{1,-1}\rangle &= \frac{1}{2}(|+z\rangle|+z\rangle + |-z\rangle|-z\rangle \\ &\quad +|+z\rangle|-z\rangle - |-z\rangle|+z\rangle) \\ &= \frac{1}{\sqrt{2}}(|+z\rangle|+x\rangle - |-z\rangle|-x\rangle) \\ &= \frac{1}{\sqrt{2}}(|-x\rangle|+z\rangle + |+x\rangle|-z\rangle) \end{aligned}$$

$$\begin{aligned} |\chi_{-1,1}\rangle &= \frac{1}{2}(|+z\rangle|+z\rangle + |-z\rangle|-z\rangle \\ &\quad -|+z\rangle|-z\rangle + |-z\rangle|+z\rangle) \\ &= \frac{1}{\sqrt{2}}(|+z\rangle|-x\rangle + |-z\rangle|+x\rangle) \\ &= \frac{1}{\sqrt{2}}(|+x\rangle|+z\rangle - |-x\rangle|-z\rangle). \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} Z_1X_2|\chi_{1,-1}\rangle = +|\chi_{1,-1}\rangle, X_1Z_2|\chi_{1,-1}\rangle = -|\chi_{1,-1}\rangle \\ Z_1X_2|\chi_{-1,1}\rangle = -|\chi_{-1,1}\rangle, X_1Z_2|\chi_{-1,1}\rangle = +|\chi_{-1,1}\rangle \end{aligned} \tag{3.11}$$

From (3.9) and (3.11) one sees that $|\psi_1\rangle$ is a linear combination of exactly those joint eigenstates of Z_1X_2 and X_1Z_2 for which the respective eigenvalues are opposite, which means, of course, that in a joint measurement the two observables will always be found to be different. With Eq. (3.6) in mind, this implies that the ensemble described by $|\psi_1\rangle$ cannot be described by any non-contextual hidden variable theory.

Note that one would already have a contradiction if quantum mechanics only predicted that the observed values of Z_1X_2 and X_1Z_2 are sometimes different, but in fact the result is even stronger, with QM and NCHV predicting exactly opposite results.

According to the argument presented in the previous paragraph, an experimental test of non-contextuality can be performed in the following way: (i) Show that $Z_1Z_2 = 1$ and $X_1X_2 = 1$ for systems prepared in a certain way. (ii) Determine whether Z_1X_2 and X_1Z_2 are equal for such systems. Note that in steps (i) and (ii) the observables Z_1, X_1, Z_2, and X_2 appear in two different contexts.

Quantum mechanics predicts that step (i) can be accomplished by constructing a source of systems described by the state $|\psi_1\rangle$ and measuring Z_1Z_2 and X_1X_2 on these systems. According to QM, both Z_1Z_2 and X_1X_2 will always be found to be equal to $+1$. This can e.g. be verified by measuring the pairs Z_1 and Z_2 and X_1 and X_2 separately on many systems, and obtaining the values of Z_1Z_2 and X_1X_2 by multiplication. Alternatively, one could also perform joint measurements of Z_1Z_2 and X_1X_2 on individual systems, but for step (i) such joint measurements are not strictly necessary. On the other hand, step (ii) definitely requires a joint measurement of Z_1X_2 and X_1Z_2, because both negative and positive values are to be expected for Z_1X_2 and X_1Z_2, and we have to determine whether their values are equal or opposite for individual systems.

3.3 . . . Leading to a Possible Experiment

One could consider realizing the above protocol with two particles. However, since a joint measurement of the two qubits is required it follows that locality is not an issue in the present experiment. This suggests looking for a single-particle realization for the sake of simplicity.

In our single-particle scheme, the first qubit is emulated by the spatial modes of propagation (paths) of a single spin-1/2 particle or photon, and the second qubit by its spin (or polarization) degree of freedom [97, 33]. Spin-1/2 and photon polarization are completely equivalent for our purposes. Our setup requires a source of polarized single particles, beam splitters, and Stern-Gerlach type devices. In practice, the experiment would be easiest to do with photons because all these

elements are readily available, in particular polarized single-photon states can be produced to excellent approximation via parametric down-conversion [39]. Besides their conceptual simplicity, single-photon experiments are attractive because very pure experimental conditions, in particular very high visibilities, can be achieved. Nevertheless, we will use the spin language in the sequel because it is more familiar to most physicists.

Consider a situation where the particle can propagate in two spatial modes u and d, and let $|z+\rangle, |z-\rangle$ etc. denote the particle's spin states as before. Then the state $|\psi_1\rangle$ of Eq. (3.8) is mapped onto the one-particle state

$$\frac{1}{\sqrt{2}}(|u\rangle|z+\rangle + |d\rangle|z-\rangle). \tag{3.12}$$

The observables Z_1, X_1, Z_2, X_2 are now represented by

$$\begin{aligned} Z_1 &= |u\rangle\langle u| - |d\rangle\langle d| \\ X_1 &= |u'\rangle\langle u'| - |d'\rangle\langle d'| \\ Z_2 &= |z+\rangle\langle z+| - |z-\rangle\langle z-| \\ X_2 &= |x+\rangle\langle x+| - |x-\rangle\langle x-|, \end{aligned} \tag{3.13}$$

where $|u'\rangle = \frac{1}{\sqrt{2}}(|u\rangle + |d\rangle), |d'\rangle = \frac{1}{\sqrt{2}}(|u\rangle - |d\rangle), |x+\rangle = \frac{1}{\sqrt{2}}(|z+\rangle + |z-\rangle), |x-\rangle = \frac{1}{\sqrt{2}}(|z+\rangle - |z-\rangle)$, i.e. u' and d' denote the output modes of a 50-50 beam-splitter with inputs u and d, and $|x+\rangle$ and $|x-\rangle$ are the spin eigenstates along the x direction . Clearly, Z_1 and X_1 act on the path, and Z_2 and X_2 on the spin degree of freedom.

To illustrate the physical meaning of the states and observables in our scheme, we show in Fig. 3.1 how a state such as $|\psi_1\rangle$ in the form of Eq. (3.12) can be prepared, and in Fig. 3.2 we show the devices that measure pairs of one-particle observables, such as Z_1 and Z_2.

The devices of Figs. 3.1 and 3.2 enable us to realize step (i) of the protocol described above. As for step (ii), Fig. 3.3 shows how a device performing a joint measurement of Z_1X_2 and X_1Z_2 can be built out of the building blocks of Fig. 3.2.

Instead of leading to detectors, the outputs of the device of Fig. 3.2b, which measures Z_1 and X_2, are now connected to two replicas of the device of Fig. 3.2c, which measure X_1 and Z_2. That the device indeed performs a joint measurement of Z_1X_2 and X_1Z_2 can be demonstrated by analyzing how it acts on the joint eigenstates of these two observables.

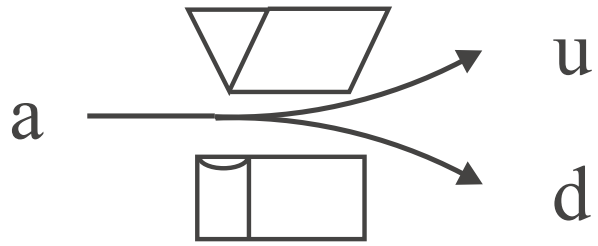
Comparison with Fig. 3.2 shows that the first device separates the two eigenspaces of the degenerate product observable Z_1X_2. Eigenstates of Z_1X_2 with eigenvalue +1 are sent up, those with eigenvalue -1 are sent down. It is important to note that this is the only way in which *eigenstates* of Z_1X_2 are affected by the first device,

i.e. they have exactly the same form in terms of the two spatial modes leading to the respective subsequent X_1Z_2-measuring device as they had in terms of the modes entering the first device. One could say that the first device "almost" performs an ideal Von Neumann measurement of the observable Z_1X_2. The difference to a Von Neumann measurement lies in the fact that the superposition between states with $Z_1X_2 = 1$ and $Z_1X_2 = -1$ is not destroyed by the device but only made ineffective because the respective components of any incoming state enter completely separated subsequent devices. Detection of the particle behind one of those two subsequent devices is a Von Neumann measurement of X_1Z_2 and at the same time completes the measurement of Z_1X_2. As is evident from the structure of the device of Fig. 3.3, the measurement of X_1Z_2 is performed by measuring X_1 and Z_2 separately as in Fig. 3.2c and multiplying the values.

While any device that performs a state analysis in the basis of common eigenstates of Z_1X_2 and X_1Z_2 can be considered to perform a joint measurement of these two observables, the particular realization presented here has the merit of showing explicitly that a joint measurement of two product observables is performed, and how the information that could have been obtained in the first stage of the measurement (the values of Z_1 and X_2 separately) has to be partially erased in order to make the second stage possible.

Let us now consider what happens when a particle in the state $|\psi_1\rangle$ enters the device of Fig. 3.3. Recall from Eq. (3.9) that $|\psi_1\rangle$ is an equally-weighted superposition of two states with opposite eigenvalues of Z_1X_2. Therefore the particle has equal amplitudes for entering either of the two X_1Z_2 devices. Explicit calculation confirms that the particle can emerge only via one of those four outputs for which the values of Z_1X_2 and X_1Z_2 are *opposite*. As explained above, after it has been shown that $Z_1Z_2 = X_1X_2 = 1$ for our particle source, NCHV predict exactly the complementary set of outputs (for which Z_1X_2 and X_1Z_2 are equal). Therefore the two theories give clearly conflicting predictions for observable effects on a non-statistical level. Of course, in a real experiment visibilities are never perfect, and one would have to use some kind of inequality to rigorously establish the contradiction. (cf. [47]).

The present scheme allows the simplest non-statistical experimental test of non-contextuality that is known to us. For a single-photon experiment that implements a statistical test of NCHV versus QM see [66]. Similarly to the original Kochen-Specker paradox it requires only a single particle (though two degrees of freedom). With the experimental setup consisting of a simple interferometer, it shows particularly clearly that the appearance of the paradox is related to the superposition principle.



$$|a\rangle|x+\rangle \rightarrow \frac{1}{\sqrt{2}}(|u\rangle|z+\rangle + |d\rangle|z-\rangle)$$

Figure 3.1: Possible way of creating the single-particle version of $|\psi_1\rangle$ given in Eq. (3.12) using a standard Stern-Gerlach apparatus. A single particle with spin state $|x+\rangle = \frac{1}{\sqrt{2}}(|z+\rangle + |z-\rangle)$, i.e. spin along the positive x direction, comes in from the left (spatial mode $|a\rangle$). By the Stern-Gerlach device, which separates incoming states according to the z-components of their spin, this is transformed into the desired superposition state. The outputs u and d could be connected to the inputs of the devices of Figures 3.2 or 3.3

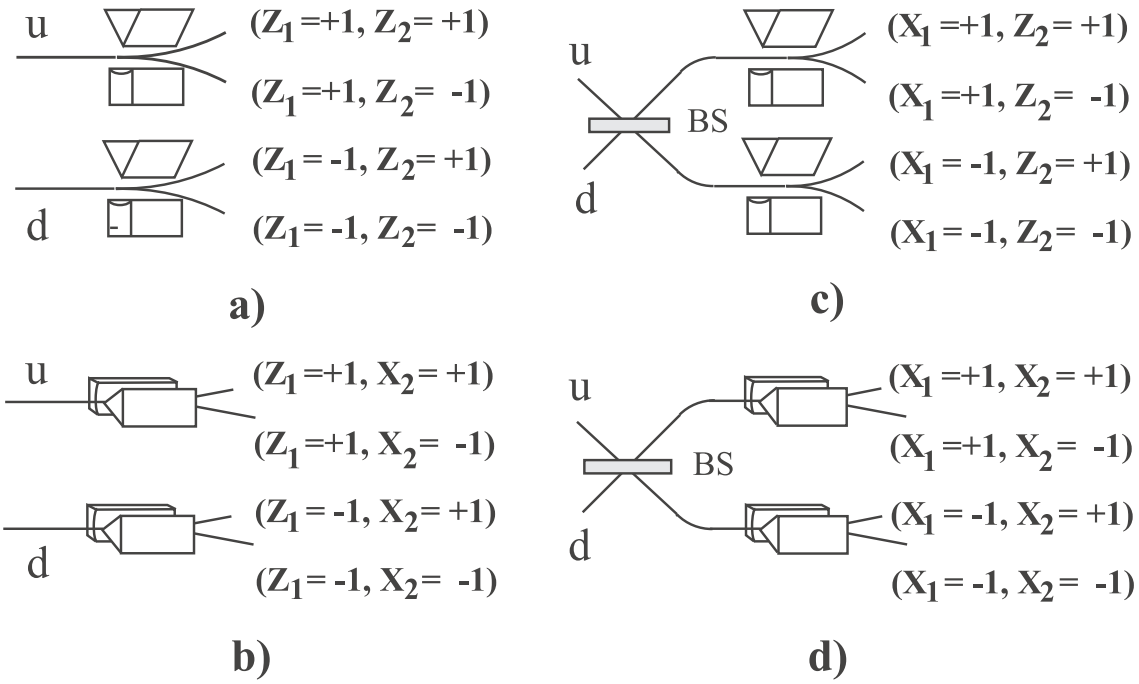


Figure 3.2: Devices for measuring pairs of the single-particle observables of Eq. (3.13). A particle comes in from the left. Note that in general the incoming states will have components in both spatial modes u and d and of different spin. The devices shown measure: a) Z_1 and Z_2; b) Z_1 and X_2; c) X_1 and Z_2; d) X_1 and X_2. BS in c) and d) stands for a $50-50$ beam-splitter (see main text), which changes the basis of path analysis from $|u\rangle, |d\rangle$, corresponding to a measurement of Z_1, to $|u'\rangle, |d'\rangle$, thus leading to a measurement of X_1. In a) and c) the Stern-Gerlach apparatus are oriented along the z-axis (measurement of Z_2), in b) and d) along the x-axis (measurement of X_2).

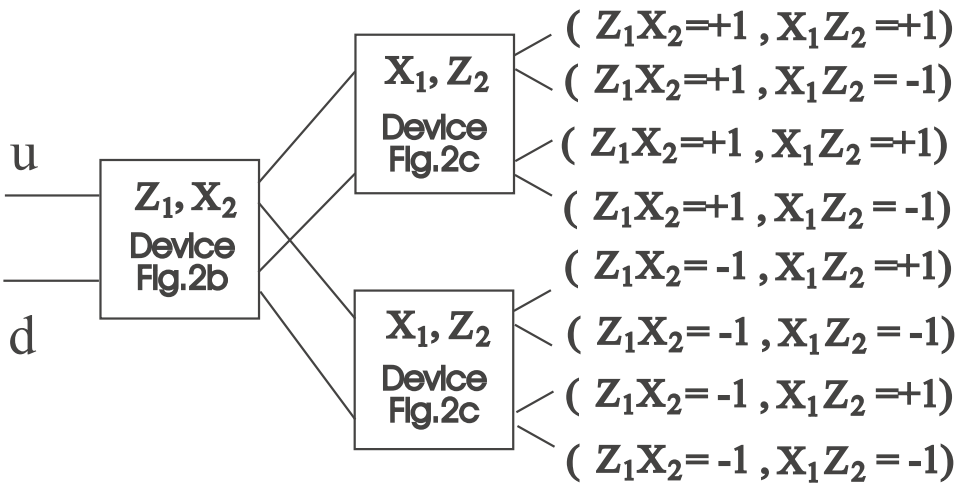


Figure 3.3: Device for performing a joint measurement of Z_1X_2 and X_1Z_2. A device performing a joint measurement of Z_1Z_2 and X_1X_2 can be constructed in an analogous way.

4 Hidden–Variable Theorems for Real Experiments

4.1 Introduction

In the original derivation of hidden–variable theorems, such as the Bell theorem, certain idealizations were made. For example the detection efficiency was originally assumed to be perfect. The case of non-unit efficiency has since been treated in detail [28, 27]. As another idealization, the precision of the measurements performed is usually not considered. When considering experimental tests of the corresponding classes of hidden–variable theories, this appears to be an important point for the following reason. An essential feature of all the hidden–variable theorems is that observables have to appear in different experimental contexts in order for a contradiction to be obtained (i.e. observables have to be measured simultaneously with different mutually exclusive observables).

For example, as we have seen above, the Kochen–Specker theorem concerns trying to assign values to all directions on the sphere subject to a constraint for triads of orthogonal directions. One can only arrive at a contradiction by considering several triads that have at least some directions in common. For Kochen-Specker experiments this implies that the observables corresponding to individual directions (i.e. the squares of the spin components along these directions) have to appear in different triads.

At first sight the usual derivations of hidden-variable theorems seem to run into problems when the finite precision of real experiments is taken into account, because then it seems impossible to ascertain that the *same* observable is really measured more than once in different experimental contexts. This question seems to be of particular relevance for the Kochen-Specker theorem in view of recent claims by Meyer that this theorem is "nullified" when the measurements have only finite precision [65].

This claim was based on the fact that it is possible to assign values to all rational directions of the sphere, which constitute a dense subset of all directions. This construction was generalized by Kent [56]. Meyer argued that, since by measurements with finite precision one cannot discriminate a dense subset from its closure, this

implies that non-contextual hidden variables cannot be excluded by any real experiment of the Kochen-Specker type. However, Meyer did not construct an explicit non-contextual hidden-variable model for real experiments with finite precision.

In the following we show how these questions can be resolved by providing a general method for the derivation of hidden-variable theorems for real experiments. In order to achieve this the concept of observable has to be changed in such a way that it has an *operational* meaning. For concreteness, imagine that an observer wants to perform a measurement of the spin square along a certain direction $\vec{n}$. There will be a certain experimental procedure for trying to do this as accurately as possible. We will refer to this procedure by saying that he sets the "control switch" of his apparatus to the position $\vec{n}$. In all experiments that we will discuss only a finite number of different switch positions is required. By definition different switch positions are clearly distinguishable for the observer, and the switch position is all he knows about. Therefore, in an operational sense the measured physical observable is entirely defined by the switch position. From the above definition it is clear that the same switch position can be chosen again and again in the course of an experiment.

In general one has to allow for the possibility that the switch position $\vec{n}$ does not uniquely determine the physical state of the measuring apparatus, i.e. there may be (hidden) properties of the apparatus over which the observer does not have full control but which may influence the result of any given measurement. Following the philosophy of deterministic hidden variable theories, one therefore has to assume that the result of any measurement will be determined not only by the hidden properties of the system, but also by those of the measuring apparatus.

In the present paper we do not discuss stochastic hidden variable theories explicitly. This does not limit the generality of the results derived because the existence of a stochastic hidden variable model for a given physical system implies that also an underlying deterministic model can be constructed which reproduces the probabilities of the stochastic model. Therefore e.g. ruling out all possible non-contextual deterministic hidden-variable models implies ruling out all possible non-contextual stochastic models as well.

4.2 Kochen–Specker Theorem for Real Experiments

As a concrete application of the ideas expressed in the two preceding paragraphs, we are now going to show how non-contextual hidden variables can be excluded by real experiments. Let us note that local hidden variables can be ruled out using an equivalent approach.

In the original Kochen-Specker situation one considers a spin-1 particle. In the ideal case of perfect precision, the relevant observables are the squares of the spin components, denoted by $S^2_{\vec{n}}$ for arbitrary directions $\vec{n}$. For a spin-1 particle one has

$$S^2_{\vec{n}_1} + S^2_{\vec{n}_2} + S^2_{\vec{n}_3} = 2 \tag{4.1}$$

for every triad of orthogonal directions $\{\vec{n}_1, \vec{n}_2, \vec{n}_3\}$. As the possible results for every $S^2_{\vec{n}_i}$ are 0 or 1, this implies that in the ideal case for every measurement of three orthogonal spin squares two of the results will be equal to one, and one of them will be equal to zero.

Let us emphasize that in our approach the operational observables are defined by the switch positions (i.e. by the best effort and knowledge of the experimenter) and therefore are not exactly identical to the exact quantum mechanical observables. In the following the symbol $S^2_{\vec{n}}$ will denote the operational observable defined by the switch position $\vec{n}$, and the term direction will be used as a synonym for switch position.

In a deterministic hidden variable theory one assumes that for every individual particle the result of the measurement of any observable $S^2_{\vec{n}}$ is predetermined by hidden properties. In *non-contextual* hidden variable theories it is furthermore assumed that this predetermined result does not depend on the "context" of the measurement, in particular which other observables are measured simultaneously with $S^2_{\vec{n}}$, but only on the switch position $\vec{n}$ and the hidden variables.

In the ideal case one could define non-contextuality in such a way that the predetermined value of some quantum mechanical observable X is required to be independent of the simultaneously measured observables only if they exactly commute with X. Note that only in the ideal case the observables corresponding to precise directions would have an operational meaning. It is evident that this weaker form of non-contextuality can only be tested in the idealized case of infinite precision.

In general the result may depend both on the hidden properties of the system and of the apparatus. Let us denote the hidden variables of the system by λ and those of the apparatus by μ. For further use, let us denote the ensemble of all possible pairs (λ, μ) by Λ. As explained above, the philosophy of non-contextual hidden variables implies the existence of a function $S^2_{\vec{n}}(\lambda, \mu)$ taking values 0 and 1 which describes the result of a measurement with switch position $\vec{n}$ on a system characterized by λ with an apparatus characterized by μ. For fixed λ and μ this function therefore assigns a value 0 or 1 to the switch position $\vec{n}$. Let us note that the models discussed by Clifton and Kent [30] are not non-contextual in the present sense because in these models the result of a measurement of $S^2_{\vec{n}}$ in general does not only depend on λ, μ, and $\vec{n}$, but also on the other observables measured simultaneously.

A Kochen–Specker experiment can now be performed by testing the validity of Eq.

(4.1) for a judiciously chosen set of triads of directions. Therefore the apparatus is required to have three switches where the three directions of a given triad can be chosen. Because the switch positions do not correspond to the ideal quantum mechanical observables the sum of the three results will not always be equal to 2. Nevertheless a contradiction between non-contextuality and quantum mechanics can be obtained in the following way.

From the Kochen-Specker theorem it follows that there are finite sets of triads for which no value assignment consistent with Eq. (4.1) is possible [58, 74]. Let us choose such a Kochen-Specker set of triads

$$\{\{\vec{n}_1, \vec{n}_2, \vec{n}_3\}, \{\vec{n}_1, \vec{n}_4, \vec{n}_5\}, ..., \}\,. \tag{4.2}$$

Let us emphasize that at least some of the switch positions $\vec{n}_i$ have to appear in several of the triads (clearly otherwise there could be no inconsistency). Let us denote the number of triads in the Kochen-Specker set (4.2) by N. The set is constructed in such a way that if one can show for some fixed λ and μ that

$$S^2_{\vec{n}_i}(\lambda,\mu) + S^2_{\vec{n}_j}(\lambda,\mu) + S^2_{\vec{n}_k}(\lambda,\mu) = 2 \tag{4.3}$$

is valid for $N-1$ of the triads $\{\vec{n}_i, \vec{n}_j, \vec{n}_k\}$, one obtains the prediction that it has to be violated for the final triad.

Suppose that for the first triad $\{\vec{n}_1, \vec{n}_2, \vec{n}_3\}$ in the Kochen-Specker set one finds that the sum of the results is equal to 2 in a fraction greater than $1-\epsilon$ of all cases. For the hidden variables this implies that

$$S^2_{\vec{n}_1}(\lambda,\mu) + S^2_{\vec{n}_2}(\lambda,\mu) + S^2_{\vec{n}_3}(\lambda,\mu) = 2 \tag{4.4}$$

for all $(\lambda,\mu) \in \Lambda_1$, where Λ_1 is some subset of the set of all hidden variables Λ with measure $p(\Lambda_1) \geq 1-\epsilon$ (by definition $p(\Lambda) = 1$). Suppose furthermore that one establishes in the same way for the second triad $\{\vec{n}_1, \vec{n}_4, \vec{n}_5\}$ that

$$S^2_{\vec{n}_1}(\lambda,\mu) + S^2_{\vec{n}_4}(\lambda,\mu) + S^2_{\vec{n}_5}(\lambda,\mu) = 2 \tag{4.5}$$

for all $(\lambda,\mu) \in \Lambda_2$ with $p(\Lambda_2) \geq 1-\epsilon$ where in general Λ_2 is a different subset of Λ, and so on for all $N-1$ triads except the final one.

This implies that for all (λ,μ) in the intersection of sets $\Lambda_\cap := \Lambda_1 \cap \Lambda_2 \cap ... \cap \Lambda_{N-1}$ the sum of results is equal to 2. Consequently, because of the structure of the Kochen-Specker set the sum of the results for the final triad has to be different from 2 (i.e. 0, 1 or 3) for all pairs $(\lambda,\mu) \in \Lambda_\cap$. This leads to the experimental prediction that the sum of results will be different from 2 for the final triad in a fraction $p(\Lambda_\cap)$ of all cases. From the property of sub-additivity ($p(\cup_i A_i) \leq \sum_i p(A_i)$) of the measure p it immediately follows that

$$p(\Lambda_\cap) \geq 1 - (N-1)\epsilon. \tag{4.6}$$

Therefore in order to experimentally disprove non-contextual hidden variables one only needs to show that the sum of results is equal to 2 in a fraction of all cases that is greater than $(N-1)\epsilon$.

If we assume for simplicity that ϵ is defined such that the fraction of "correct" (equal to 2) results is larger than $1-\epsilon$ for all triads (including the final one) then the above results allow us to derive a bound on the size of the experimental imperfection ϵ such that an experimental contradiction with non-contextuality can still be obtained: ϵ has to be smaller than $1/N$. Note that ϵ describes all the imperfections of a real experiment including finite precision but also e.g. imperfect state preparation and non-unit detection efficiency. The value of N and therefore of the bound on ϵ depends on the particular Kochen-Specker set used [58, 74].

As we have already noted above, an inevitable requirement for the contradiction to be obtained is the fact that the function $S^2_{\vec{n}_1}(\lambda,\mu)$, or in general functions corresponding to at least some switch positions, appear in more than one out of the N triads. This appearance of the same function in different lines of the mathematical proof (corresponding to different experimental contexts) is possible in spite of finite experimental precision only because we defined our observables operationally via the switch positions.

We have shown how non-contextual hidden-variable theories can be disproved by real experiments. This clarifies questions raised by [65]. In view of our results, we would assert that the Kochen-Specker theorem is not "nullified" by finite measurement precision. Let us note that independent arguments in favor of this conclusion were given in [63, 3, 2, 60]. Our suggestion how to perform a Kochen-Specker experiment was inspired by some of Mermin's remarks in [63].

Using the same method one can also show that local hidden variables can be disproved in real experiments, e.g. using the GHZ [47] form of Bell's theorem which is also based on sets of propositions that cannot be consistently satisfied by hidden variables. Inequalities analogous to Eq. (4.6) can be derived and tested experimentally [73].

4.3 Hidden Variables: Perspectives

In the previous chapter we have presented a very simple Kochen–Specker type argument. It seems unlikely that much further simplification is possible. In this chapter we have analyzed the derivation of hidden–variable theorems for real experimental conditions. We have seen that the theorems, including those on non–contextuality, are robust under real–world conditions and thus experimentally testable.

Let us emphasize that 36 years after Bell (and 68 years after von Neumann) there are still interesting open questions in the field of hidden variables. Most importantly, on the experimental side, a loophole–free demonstration of the violation of Bell's inequalities is still missing. Such an experiment would require both space-like separation of the measurements performed on each entangled pair and high detection efficiency. For the realization of such experiments new ways of establishing contradictions between local hidden variables and quantum mechanics may be helpful. E.g. Eberhard [38] showed that the required detection efficiencies are lower if non–maximally entangled states together with appropriate analyzer settings are used. A possible new approach would be to consider adaptive measurements [78], or in general joint measurements on several pairs.

On the theoretical side, there are many connections between the study of general Bell's inequalities and the classification and quantification of entanglement, cf. e.g. [80]. A particularly interesting open question is whether the so–called "bound entangled" states [51], which are states from which no maximal entanglement can be distilled, admit local hidden variable models, i.e. whether there is entanglement without non–locality.

One may also hope that a detailed understanding of the quantum weirdness, i.e. the differences between quantum mechanics and the classical world view, might help to generate ideas how to exploit it in order to perform tasks that are classically impossible.

Conclusions and Outlook

During the three years of my PhD studies I had the good fortune to come into contact with many fields, some of which are not even mentioned in this thesis. Quite naturally I learned about various sub-fields of quantum information, ranging from cloning and quantum cryptography, over quantum computing, to the study of entanglement and its purification. I got to know quantum optics, another field which had not figured in my undergraduate studies, from the experimental and also somewhat from the theoretical side. I learned a lot about hidden-variable theorems and the related experiments and some basic but important facts about the practical aspects of decoherence, mostly from my experimental colleagues. Towards the end I was glad to learn some things about entanglement in quantum field theory, a topic I liked because it created a bridge to my earlier studies. I also found that my expectations had been correct: indeed there were many opportunities to discuss, learn and think about the basic questions of quantum physics, together with Anton Zeilinger, Časlav Brukner, and many others.

One of the most important things that I learned is that it is very nice and that it can also be quite fruitful for a theorist to be in close (in my case: permanent) contact with an experimental group, especially if it is such a good one. Not least because your knowledge of physics is constantly tested by the questions of your experimental friends, which have a tendency to always be related to the real world. I learned other things which I think will be useful, such as writing a proposal, organizing a workshop. Once, we even made a movie. All this was usually done in a team of great people, from about ten different nations. So much for myself.

What have we learned from quantum information in general? Most importantly, that, with the help of quantum physical systems, one can do things that are unthinkable classically, starting of course with the discovery of Bell's inequalities. In the last years we have also learned a lot about what can be done in practice in the lab. This includes many amazing things, such as multi-particle entanglement [72], the interference of large molecules [4], the study of single ions in traps coupled via single phonons [69], and single photons interacting with single atoms in cavities of incredibly high quality [49]. It is probably also fair to say that quantum information has led to a new way of looking at physics, for example we see entanglement almost everywhere.

Let me try to summarize the major challenges for quantum information at the present stage by two questions: What else could we do (with quantum systems), and what can we really build? The first question has a theoretical, the second an experimental flavor, but people from both sides are trying to find answers to both. A natural way of attacking the first one is to look for new quantum algorithms that outperform classical ones. Physicists and computer scientists are also investing a lot of effort into trying to prove general results on the power of quantum computation. One may also feel that "quantum non-locality" has not yet been fully exploited. There must be more that one can do with distributed entanglement than violate Bell's inequalities and perform cryptography and teleportation. One promising result is the reduction of communication complexity [29].

As for the second question, the main goals are to build a quantum computer of serious size and to achieve quantum communication over long distances. Hope rests on the continuous improvement of solid state techniques [55], for example for semiconductors and super-conductivity, and also in new technological achievements, such as Bose-Einstein condensation [1] and the development of laser cooling [64]. The main difficulty in designing and building a quantum computer is to keep quantum coherence, where normally it is lost very fast. This is attempted using a combination of technological approaches such as cooling, isolation, the use of systems which have low decoherence by nature, and algorithmic methods, such as quantum error correction and fault-tolerant computing.

Thus, there is a close connection between quantum computing and a more foundationally oriented research programme, the preparation and study of larger and larger superposition states, "Schrödinger cats". One of the driving hopes behind such a programme was formulated in the preface to this thesis: something new could turn up. We might find the limits of validity of quantum physics, maybe even something like the border of the classical world. At present such hopes, although no strangers to the author's heart, seem preposterous if not unreasonable. One argument in their favor which has some appeal for the author is that usually in physics linearity is an approximation. The connection between the linearity of quantum mechanics and special relativity should be kept in mind in this context, but it is well known that non-linear modifications of quantum physics are conceivable if some of its basic assumptions are given up.

There are other more modest hopes for the future of quantum information. There is little doubt that we will learn a lot more about physics in a practical and quantitative sense. We also hope for new conceptual insights, for example concerning the relation of quantum and classical information. Quantum information should meet other areas of physics besides quantum optics, most prominently statistical physics and quantum field theory. New interesting physics should come out of such encounters. A more ambitious hope again is that the concept of information may help us to arrive at a

deeper understanding of the basic principles of quantum physics [16, 17, 94].

Personally I hope to use many of the things that I have learned and to continue working in a fascinating field dealing with fundamental questions and yet close to the real world of experiments.

Papers by the Author

1. G. Weihs, T. Jennewein, C. Simon, H. Weinfurter, and A. Zeilinger: *Violation of Bell's inequality under strict Einstein locality conditions.* Phys. Rev. Lett., **81**, 5039-43 (1998), quant-ph/9810080

2. C. Simon, G. Weihs, and A. Zeilinger: *Quantum Cloning and Signaling.* Acta Phys. Slov., **49**, 755-760 (1999)

3. C. Simon, G. Weihs, and A. Zeilinger: *Optimal quantum cloning and universal NOT without quantum gates,* J. Mod. Opt., **47**, 233-246 (2000)

4. C. Simon, G. Weihs, and A. Zeilinger: *Optimal Quantum Cloning via Stimulated Emission,* Phys. Rev. Lett., **84**, 2993 (2000), quant-ph/9910048

5. T. Jennewein, C. Simon, G. Weihs, H. Weinfurter, and A. Zeilinger: *Quantum Cryptography with Polarization Entangled Photons,* Phys. Rev. Lett., **84**, 4729 (2000), quant-ph/9912117

6. J. Kempe, C. Simon, and G. Weihs: *Optimal Photon Cloning,* Phys. Rev. A **62**, 032302 (2000), quant-ph/0003025

7. C. Simon, M. Żukowski, H. Weinfurter, and A. Zeilinger: *A feasible "Kochen-Specker" experiment with single particles,* Phys. Rev. Lett., **85**, 1783 (2000), quant-ph/0009074

Bibliography

[1] M.H. Anderson, J.R. Ensher, M.R. Matthews, C.E. Wieman, and E.A. Cornell, Science **269**, 198 (1995).

[2] D.M. Appleby, quant-ph/0005010.

[3] D.M. Appleby, quant-ph/0005056.

[4] M. Arndt, O. Nairz, J. Voss-Andreae, C. Keller, G. van der Zouw, and A. Zeilinger, Nature **401**, 680 (1999).

[5] A. Aspect, J. Dalibard, and G. Roger, Phys. Rev. Lett. **49**, 1804 (1982).

[6] H. Bechmann-Pasquinucci and N. Gisin, Phys. Rev. A **59**, 4238 (1999).

[7] J.S. Bell, Physics (Long Island City, N.Y.) 1, 195 (1964). Reprinted in [9].

[8] J.S. Bell, Rev. Mod. Phys. **38**, 447 (1966).

[9] J.S. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge Univ. Press, Cambridge, 1987).

[10] C.H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).

[11] C.H. Bennett, G. Brassard, S. Briedbart, and S. Wiesner, Advances in Cryptology: Proceedings of Crypto '82 (Plenum, New York), 267 (1983).

[12] C.H. Bennett and D.P. DiVincenzo, Nature **404**, 247 (2000).

[13] D. Bouwmeester, J.W. Pan, K. Mattle, M. Eibl, H. Weinfurter, and A. Zeilinger, Nature **390**, 575 (1997).

[14] D. Bouwmeester, private communication.

[15] S.L. Braunstein, V. Bužek, and M. Hillery, quant-ph/0009076.

[16] C. Brukner and A. Zeilinger, Phys. Rev. Lett. **83**, 3354 (1999).

[17] C. Brukner, *Information in individual quantum systems* (PhD thesis, Technical University of Vienna, 1999).

[18] D. Bruß, A. Ekert, and C. Macchiavello, Phys. Rev. Lett. **81**, 2598 (1998).

[19] V. Bužek, M. Hillery, and R.F.Werner, *Phys. Rev. A* **60**, R2626 (1999); *J. Mod. Opt.* **47**, 211 (2000).

[20] V. Bužek, S.L. Braunstein, M. Hillery, and D. Bruß, Phys. Rev. A **56**, 3446 (1997).

[21] V. Bužek and M. Hillery, Phys. Rev. A **54**, 1844 (1996).

[22] V. Bužek and M. Hillery, Phys. Rev. Lett. **81**, 5003 (1998).

[23] C. Simon and V. Bužek, in preparation

[24] A. Cabello and G. García-Alcaine, Phys. Rev. Lett. **80**, 1797 (1998).

[25] A.R. Calderbank and P. W. Shor, Phys. Rev. A **54**, 1098 (1996).

[26] J.I. Cirac and P. Zoller, Phys. Rev. Lett. **74**, 4091 (1995).

[27] J.F. Clauser and A. Shimony, Rep. Prog. Phys. **41**, 1881 (1978).

[28] J.F. Clauser and M. A. Horne, Phys. Rev. D **10**, 526 (1974).

[29] R. Cleve and H. Buhrmann, Phys. Rev. A **56**, 1201 (1997).

[30] R. Clifton and A. Kent, P. Roy. Soc. Lond. A MAT **456** 2101 (2001)

[31] M.J. Collett, Phys. Rev. A **38**, 2233 (1988).

[32] D.G. Cory et al, quant-ph/0004104.

[33] M. Czachor, Phys. Rev. A **49**, 2231 (1994).

[34] M. Czachor and M. Kuna, Phys. Rev. A **58**, 128 (1998).

[35] F. De Martini, Phys. Rev. Lett. **81**, 2842 (1998).

[36] F. De Martini and V. Mussi, Fortschr. Phys. **48** (5-7) 413 (2000)

[37] D. Dieks, Phys. Lett. **92**A, 271 (1982).

[38] P. H. Eberhard, Phys. Rev. A **47**, R747 (1993)

[39] S. Friberg, C.K. Hong, and L. Mandel, Phys. Rev. Lett. **54**, 2011 (1985).

[40] G.C. Ghirardi, A. Rimini, and T. Weber, Lett. Nuov. Cim. **27**, 293 (1980).

[41] N. Gisin, Helv. Phys. Acta **62**, 363 (1989).

[42] N. Gisin, Phys. Lett. A **143**, 1 (1990).

[43] N. Gisin, Phys. Lett. A **242**, 1 (1998).

[44] N. Gisin and S. Massar, Phys. Rev. Lett. **79**, 2153 (1997).

[45] A.M. Gleason, J. Math. Mech. **6**, 885 (1957).

[46] D.M. Greenberger, M. Horne, A. Shimony, and A. Zeilinger, Am. J. Phys. **58**, 1131 (1990).

[47] D.M. Greenberger, M. Horne, and A. Zeilinger, in *Bell's Theorem, Quantum Theory, and Conceptions of the Universe*, edited by M. Kafatos (Kluwer, Dordrecht, 1989).

[48] L.K. Grover, Phys. Rev. Lett. **79**, 325 (1997).

[49] E. Hagley, X. Maitre, G. Nogues, C. Wunderlich, M. Brune, J.M. Raimond, and S. Haroche, Phys. Rev. Lett. **79**, 1 (1997).

[50] N. Herbert, Found. Phys. **12**, 1171 (1982).

[51] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **80**, 5239 (1998).

[52] L.P. Hughston, R. Jozsa, W. K. Wootters, Phys. Lett. A **183**, 14 (1993).

[53] C. Itzykson and J.–B. Zuber, *Quantum Field Theory* (McGraw–Hill, 1985).

[54] T. Jennewein, C. Simon, G. Weihs, H. Weinfurter, A. Zeilinger, Phys. Rev. Lett **84**, 4729 (2000).

[55] B.E. Kane, Nature **393**, 133 (1998).

[56] A. Kent, Phys. Rev. Lett. **83** (1999) 3755.

[57] M. Keyl and R.F. Werner, J. Math. Phys. **40**, 3283 (1999).

[58] S. Kochen and E. P. Specker, J. Math. and Mech. **17**, 59 (1967).

[59] P.G. Kwiat, K. Mattle, H. Weinfurter, A. Zeilinger, A.V. Sergienko, and Y. Shih, Phys. Rev. Lett. **75**, 4337 (1995).

[60] J.–Å. Larsson, quant-ph/0006134.

[61] L. Mandel, Nature **304**, 188 (1983).

[62] S. Massar and S. Popescu, Phys. Rev. Lett. **74**, 1259 (1995).

[63] N.D. Mermin, quant-ph/9912081.

[64] H.J. Metcalf and P. van der Straten, *Laser Cooling and Trapping* (Springer-Verlag, New York, 1999).

[65] D. Meyer, Phys. Rev. Lett. **83**, 3751 (1999).

[66] M. Michler, H. Weinfurter, and M. Żukowski, Phys. Rev. Lett. **84**, 5457 (2000).

[67] A. Migdall, Phys. Today, January 1999, p. 41 (1999).

[68] P.W. Milonni and M.L. Hardies, Phys. Lett. **92**A, 321 (1982).

[69] C. Monroe, D.M. Meekhof, B.E. King, W.M. Itano, and D.J. Wineland, Phys. Rev. Lett. **75**, 4714 (1995).

[70] N.J. Cerf, Acta Phys. Slov. **48**, 115 (1998).

[71] D.S. Naik, C.G. Peterson, A.G. White, A.J. Berglund, P.G. Kwiat, Phys. Rev. Lett **84**, 4733 (2000).

[72] J.W. Pan, D. Bouwmeester, M. Daniell, H. Weinfurter, and A. Zeilinger, Nature **403**, 515 (2000).

[73] J.W. Pan, D. Bouwmeester, M. Daniell, H. Weinfurter, and A. Zeilinger, Nature **403**, 515 (2000).

[74] A. Peres, J. Phys. A **24**, L175 (1991).

[75] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993).

[76] S. Popescu and D. Rohrlich, Found. Phys. **24**, 379 (1994).

[77] J. Preskill, Lecture Notes on Quantum Computation, `http://www.theory.caltech.edu/people/preskill/ph229/#lecture`

[78] T. Rudolph, private communication.

[79] J.J. Sakurai, *Modern quantum mechanics* (Addison-Wesley, 1994), Section 3.8.

[80] V. Scarani and N. Gisin, private communication.

[81] P.W. Shor, Phys. Rev. A **52**, 2493 (1995).

[82] P.W. Shor, Proc. 37th Symp. on Foundations of Computer Science.

[83] P.W. Shor, SIAM J. Comp. **26**, No. 5, 1484 (1997).

[84] C. Simon, V. Bužek, and N. Gisin, in preparation.

[85] E.P. Specker, *Selecta* (Birkhäuser Verlag, Basel, 1990).

[86] A.M. Steane, Phys. Rev. Lett. **77**, 793 (1996).

[87] W. Tittel, J. Brendel, H. Zbinden, M. Gisin, Phys. Rev. Lett. **84**, 4737 (2000).

[88] Q.A. Turchette, C.J. Hood, W. Lange, H. Mabuchi, and H.J. Kimble, Phys. Rev. Lett. **75**, 4710 (1995).

[89] D.F. Walls and G.I. Milburn, *Quantum Optics* (Springer-Verlag, Berlin, 1995), Chap. 5.

[90] G. Weihs, T. Jennewein, C. Simon, H. Weinfurter, and A. Zeilinger, Phys. Rev. Lett. **81**, 5039 (1998).

[91] R.F. Werner, Phys. Rev. A **58**, 1827 (1998).

[92] W.K. Wootters and W.H. Zurek, Nature (London) **299**, 802 (1982).

[93] J. Ye, D.W. Vernooy, and H.J. Kimble, Phys. Rev. Lett. **83**, 4987 (1999)

[94] A. Zeilinger, Found. Phys. **29**, 631 (1999).

[95] A. Zeilinger, M.A. Horne, H. Weinfurter, and M. Żukowski, Phys. Rev. Lett. **78**, 3031 (1997).

[96] A. Zeilinger and M. Żukowski, to appear in Rev. Mod. Phys.

[97] M. Żukowski, Phys. Lett. A **157**, 198 (1991).

[98] W.H. Zurek, Physics Today **44**(10), October, 36 (1991).

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